Ciências
ULisboa

Image representation and processing tools
Useful 1D and 2D functions
Convolution and Integral Transforms

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Useful functions – 1D

Rectangle function

$$\mathrm{rect}(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Sinc function

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

Signum function

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Triangle function

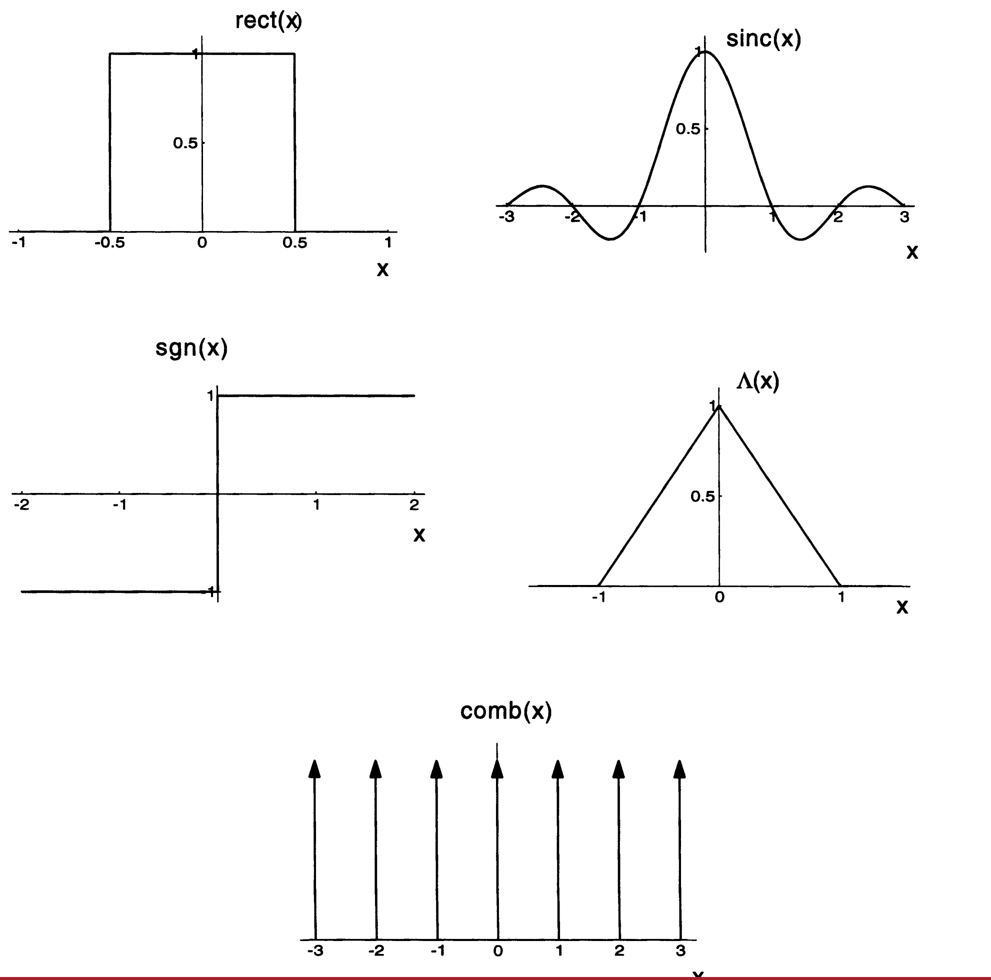
$$\Lambda(x) = \begin{cases} 1 - |x| & |x| \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Comb function

$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

Circle function

$$\mathrm{circ}(\sqrt{x^2 + y^2}) = \begin{cases} 1 & \sqrt{x^2 + y^2} < 1 \\ \frac{1}{2} & \sqrt{x^2 + y^2} = 1 \\ 0 & \text{otherwise} \end{cases}$$



Goodman J. W., **Introduction to Fourier Optics**, 3rd ed. Roberts & Company, 2005.
Gaskill, **Linear Systems, Fourier Transforms, and Optics**, Wiley, 1978

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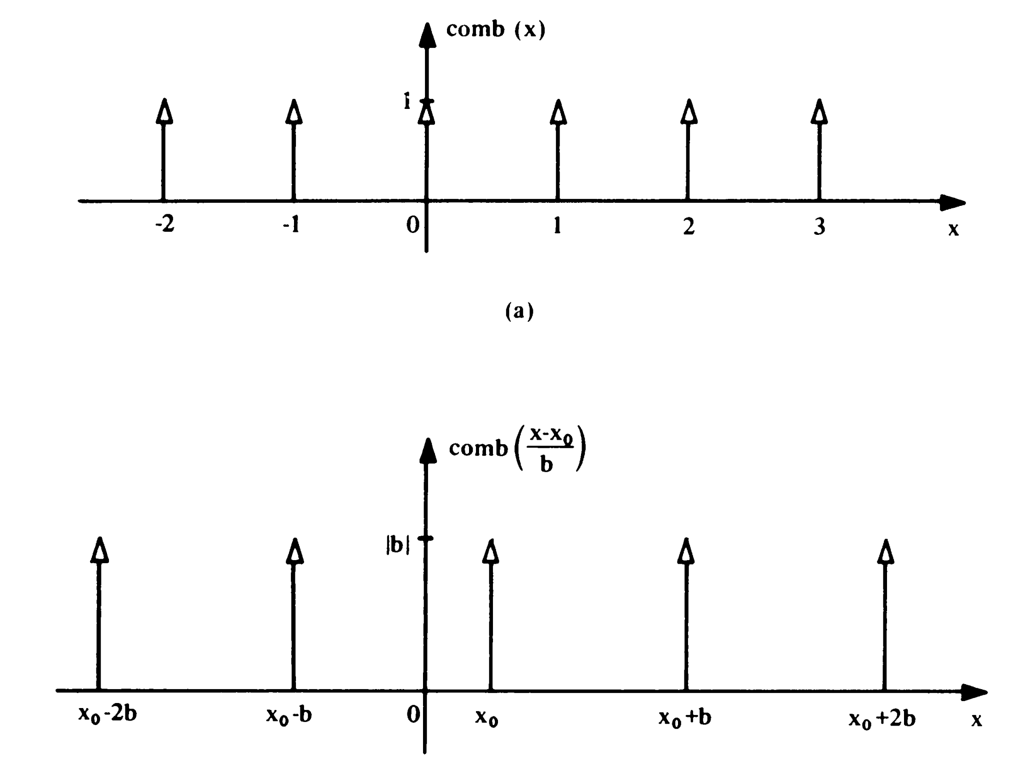
Comb function

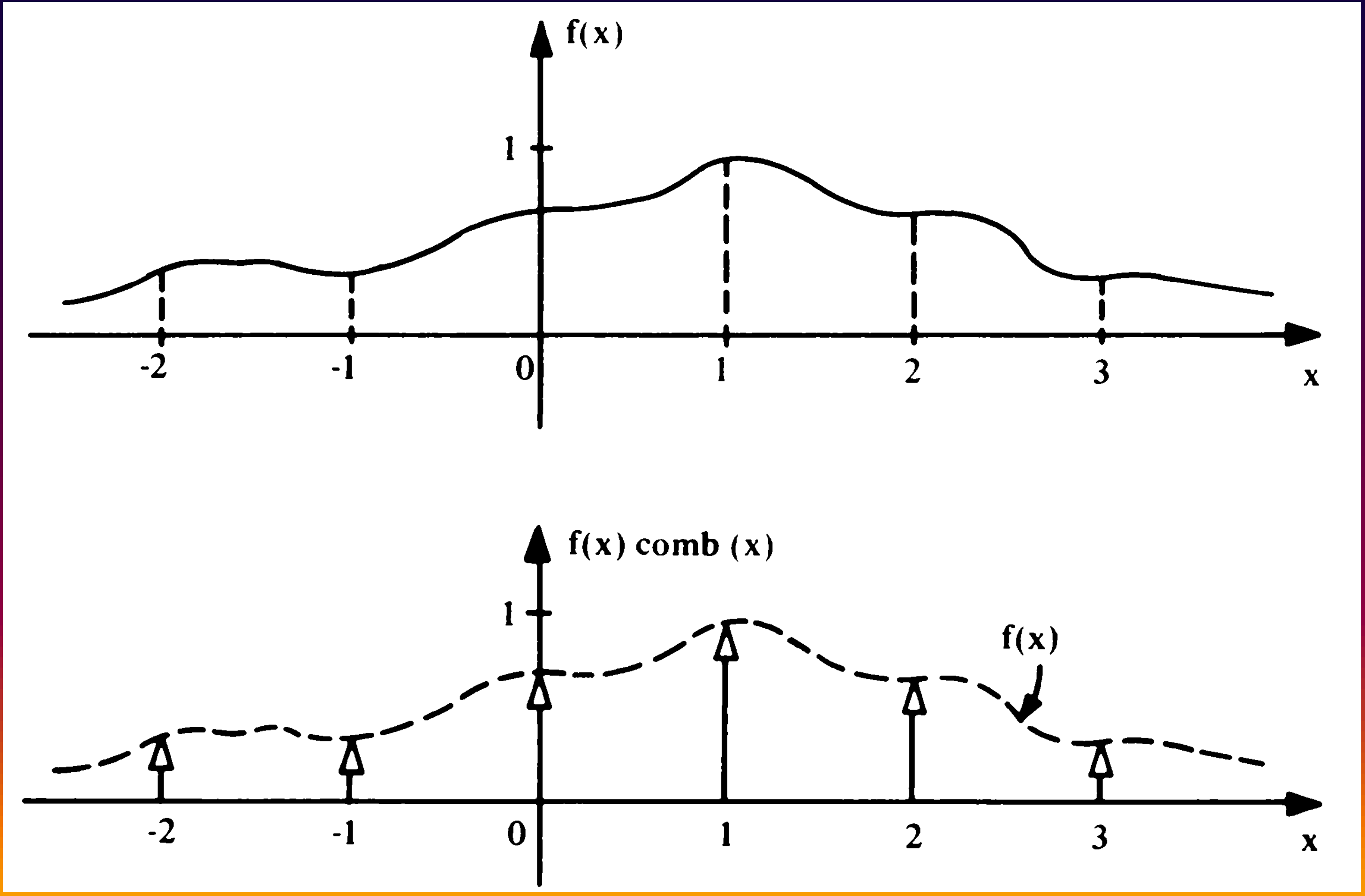
$$\mathrm{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n),$$



$$\mathrm{comb}\left(\frac{x-x_0}{b}\right) = |b| \sum_{n=-\infty}^{\infty} \delta(x-x_0-nb),$$

Comb function & sampling

$$f(x)\left[\frac{1}{|b|}\mathrm{comb}\left(\frac{x-x_0}{b}\right)\right]=\sum_{n=-\infty}^{\infty}f(x_0+nb)\delta(x-x_0-nb),$$



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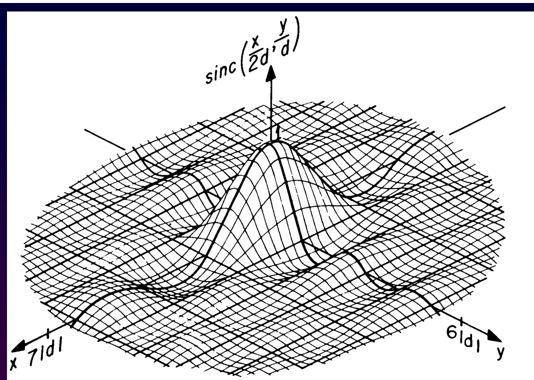
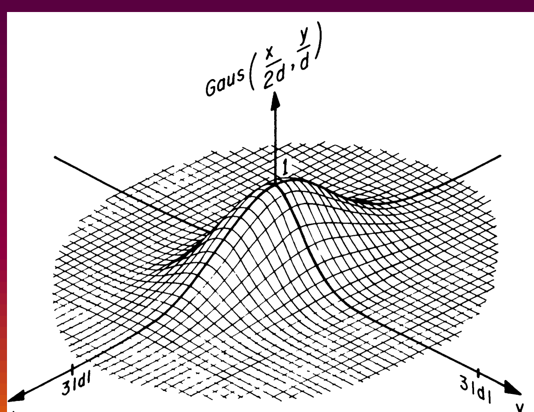
Useful 2D functions - separable

$$\mathrm{sinc}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right)=\mathrm{sinc}\left(\frac{x-x_0}{b}\right)\mathrm{sinc}\left(\frac{y-y_0}{d}\right)$$

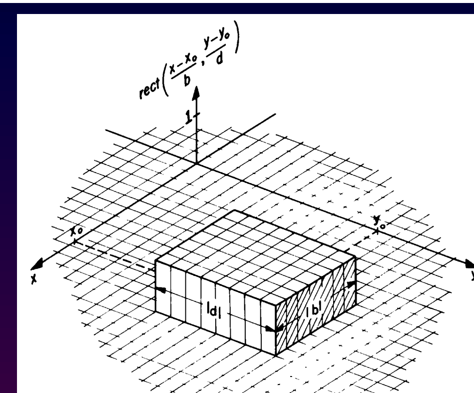


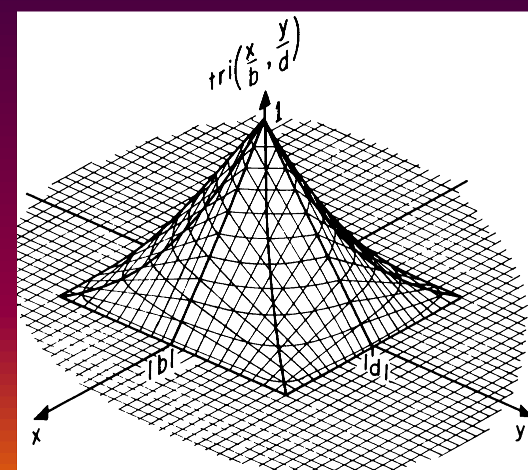
$$\mathrm{sinc}^2\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right)=\mathrm{sinc}^2\left(\frac{x-x_0}{b}\right)\mathrm{sinc}^2\left(\frac{y-y_0}{d}\right)$$



$$\mathrm{Gaus}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right)=\mathrm{Gaus}\left(\frac{x-x_0}{b}\right)\mathrm{Gaus}\left(\frac{y-y_0}{d}\right)$$

$$\mathrm{rect}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right)=\mathrm{rect}\left(\frac{x-x_0}{b}\right)\mathrm{rect}\left(\frac{y-y_0}{d}\right).$$





$$\mathrm{tri}\left(\frac{x-x_0}{b},\frac{y-y_0}{d}\right)=\mathrm{tri}\left(\frac{x-x_0}{b}\right)\mathrm{tri}\left(\frac{y-y_0}{d}\right).$$

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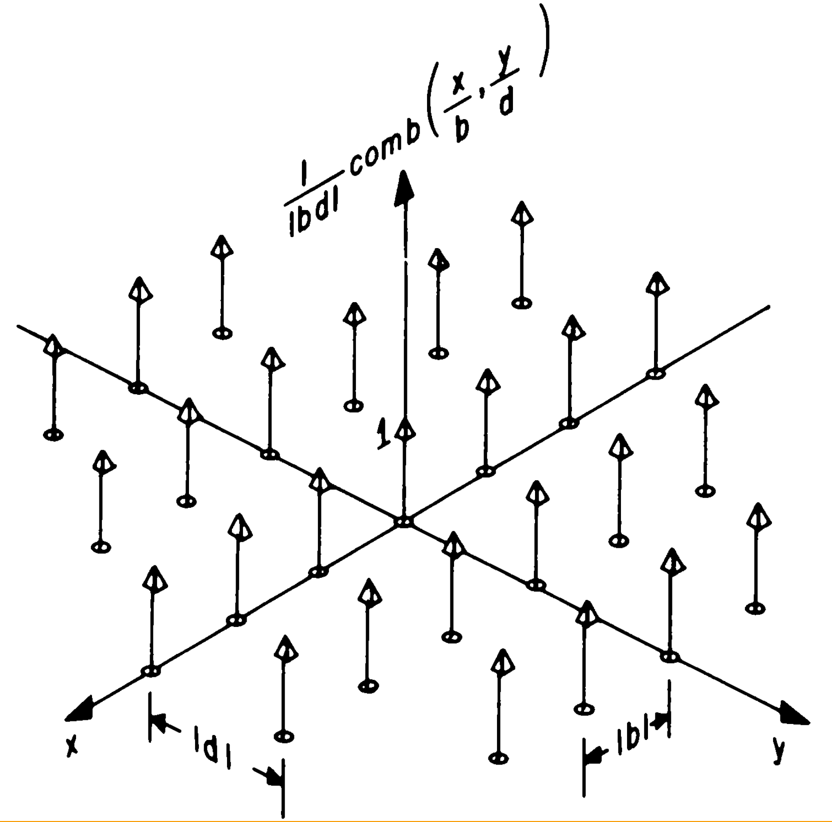
2D Comb

$$\mathrm{comb}\left(\frac{x-x_0}{b}\right)=|b|\sum_{n=-\infty}^{\infty}\delta(x-x_0-nb)$$

$$\mathrm{comb}(x,y)=\mathrm{comb}(x)\,\mathrm{comb}(y)$$

$$\frac{1}{|bd|}\mathrm{comb}\left(\frac{x}{b},\frac{y}{d}\right)=\frac{1}{|b|}\mathrm{comb}\left(\frac{x}{b}\right)\frac{1}{|d|}\mathrm{comb}\left(\frac{y}{d}\right)$$



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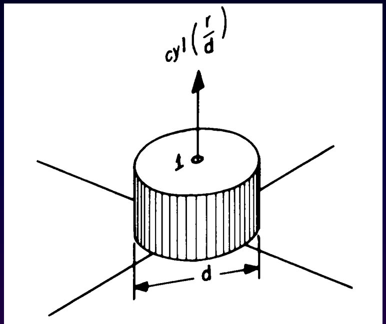
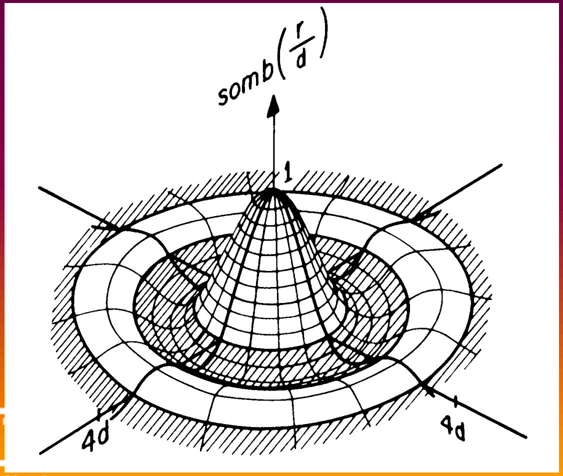
Useful 2D functions – non-separable

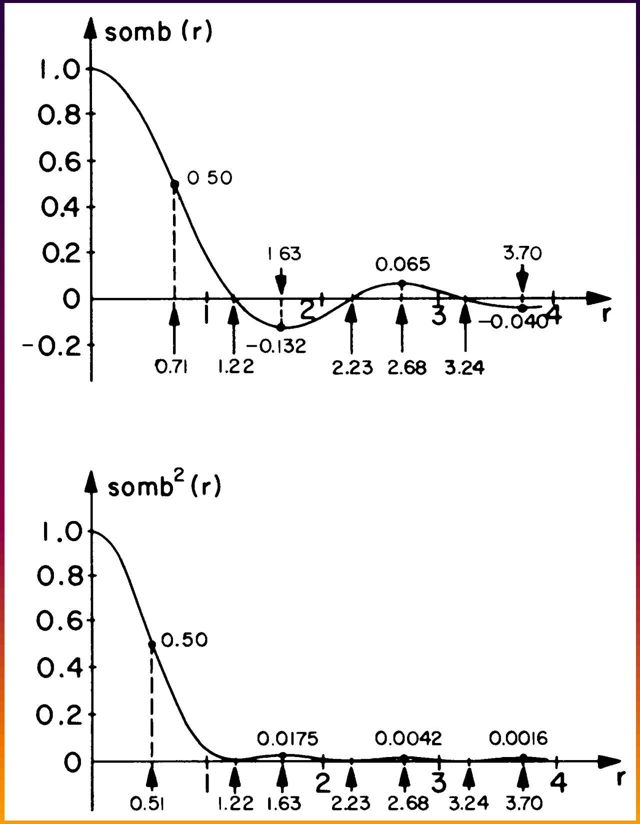
$$\mathrm{cyl}\left(\frac{r}{d}\right) = \begin{cases} 1, & 0 \leqslant r < \frac{d}{2} \\ \frac{1}{2}, & r = \frac{d}{2} \\ 0, & r > \frac{d}{2} \end{cases}$$



$$\mathrm{Gaus}\left(\frac{r}{d}\right) = e^{-\pi (r/d)^2}$$

$$\mathrm{somb}\left(\frac{r}{d}\right) = \frac{2J_1\left(\frac{\pi r}{d}\right)}{\left(\frac{\pi r}{d}\right)}$$

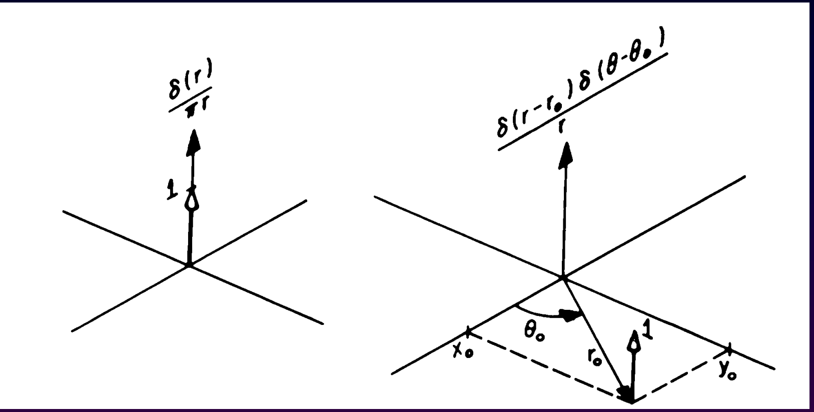




2D δ-Dirac

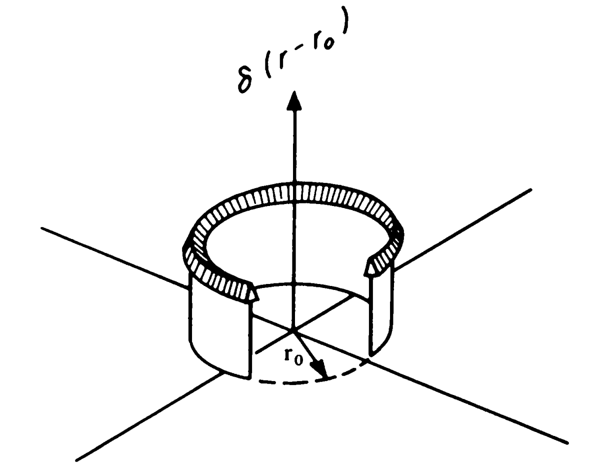
$$\delta(x-x_0, y-y_0) = \delta(x-x_0)\delta(y-y_0)$$

$$\delta(\mathbf{r}) = \frac{\delta(r)}{\pi r}$$



$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(r-r_0)}{r_0}\delta(\theta-\theta_0)$$

$$\delta(r-r_0)$$



Linearly deformed 2D functions

f (x,y) → f [w₁(x,y), w₂(x,y)] = g (x,y)

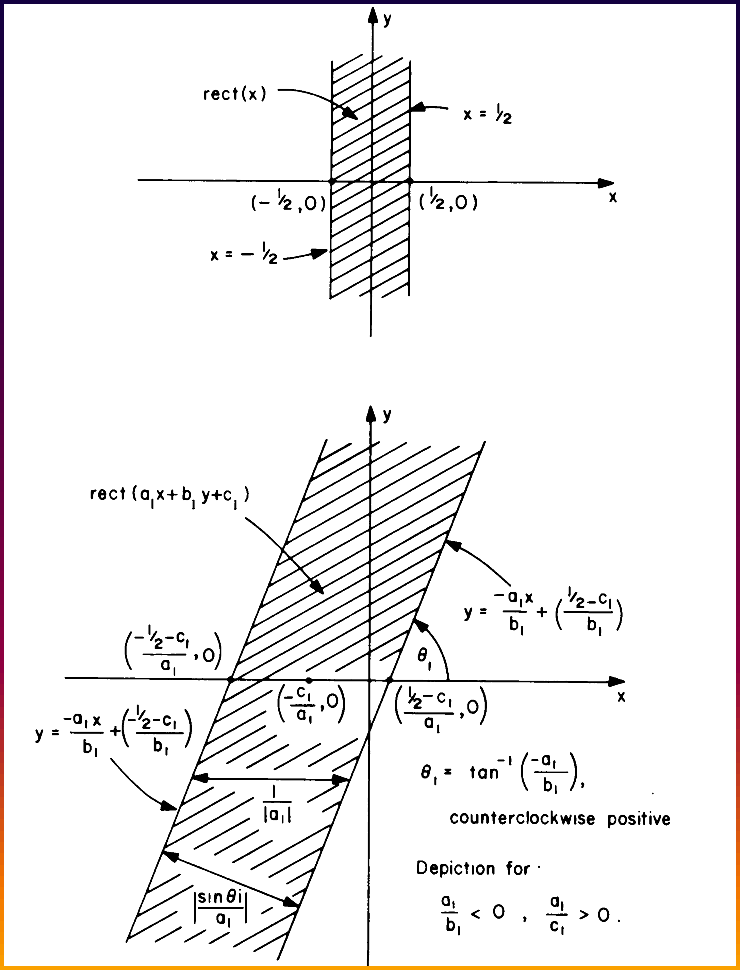
$$f(x,y) \rightarrow f[w_1(x,y),\, w_2(x,y)] = g(x,y)$$

$$w_1(x,y) = a_1 x + b_1 y + c_1$$

$$w_2(x,y) = a_2 x + b_2 y + c_2$$

Ex. 1

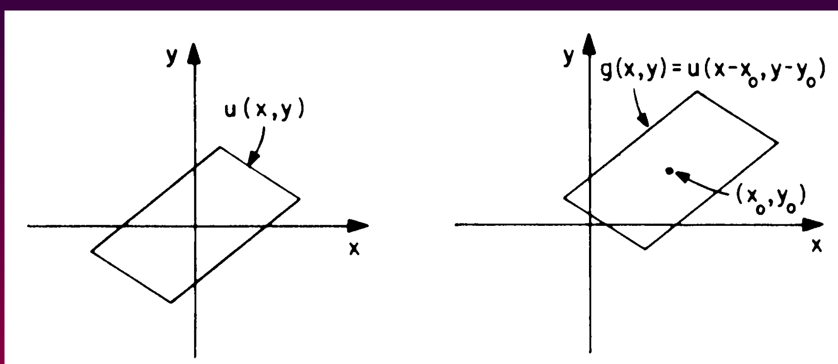
$$f(x,y) = rect(x) \rightarrow g(x,y) = rect(a_1 x + b_1 y + c_1)$$

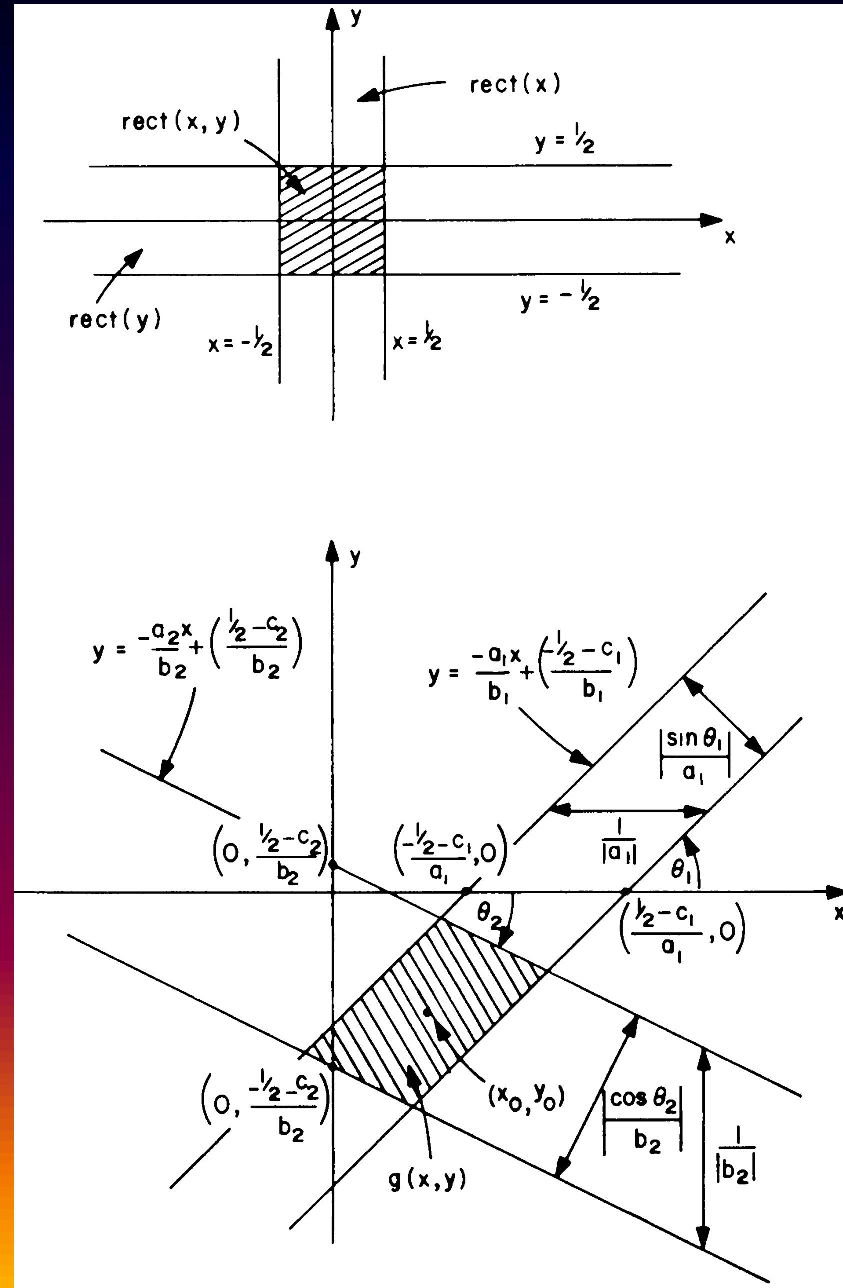


Linearly deformed 2D functions

Ex. 2

$f(x,y) = rect(x,y) \rightarrow g(x,y) =$
$rect\,(a_1x+b_1y+c_1)\;rect\,(a_2x+b_2y+c_2)$





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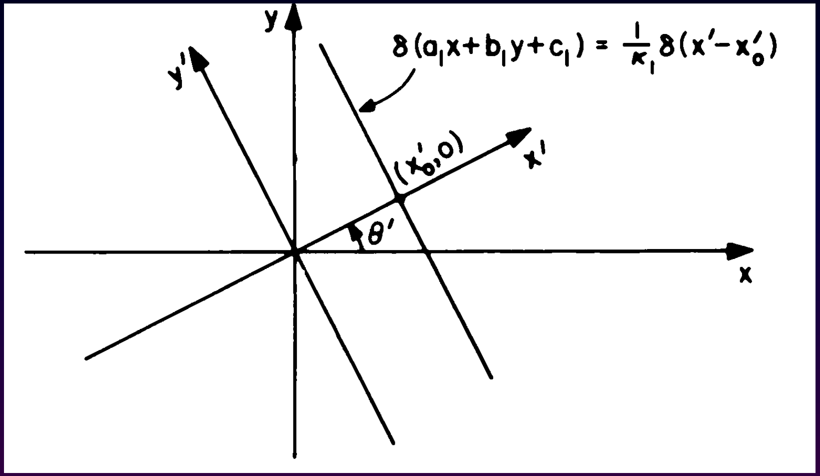
Linearly deformed 2D functions

$$\delta(a_1 x + b_1 y + c_1)$$

$$\delta(a_1 x + b_1 y + c_1) = \frac{1}{|a_1|}\delta\left(x + \frac{b_1 y}{a_1} + \frac{c_1}{a_1}\right)$$

$$= \frac{1}{|b_1|}\delta\left(y + \frac{a_1 x}{b_1} + \frac{c_1}{b_1}\right)$$



2D functions skew-periodic

$$\mathrm{comb}(a_1 x + b_1 y + c_1,\ a_2 x + b_2 y + c_2)$$

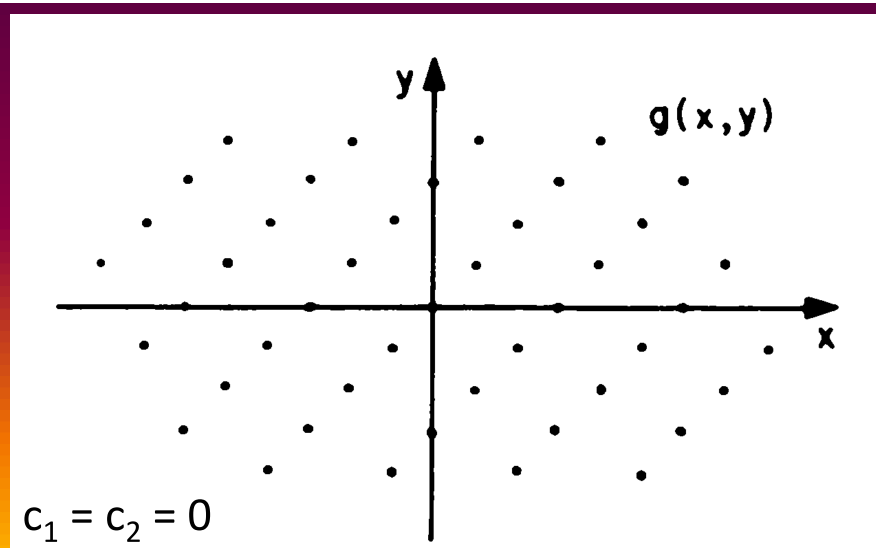
$$g(x,y) = \mathrm{comb}(a_1 x + b_2 y + c_1, a_2 x + b_2 y + c_2)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(a_1 x + b_1 y + c_1 - n)\delta(a_2 x + b_2 y + c_2 - m)$$

$$= \frac{1}{|D|} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(x - x_0 - \frac{b_2 n}{D} + \frac{b_1 m}{D}\right)$$

$$\times \delta\left(y - y_0 + \frac{a_2 n}{D} - \frac{a_1 m}{D}\right), \qquad (3$$

D = determinant of matrix $[a_1\ b_1; a_2\ b_2]$



2D arbitrary regular sampling

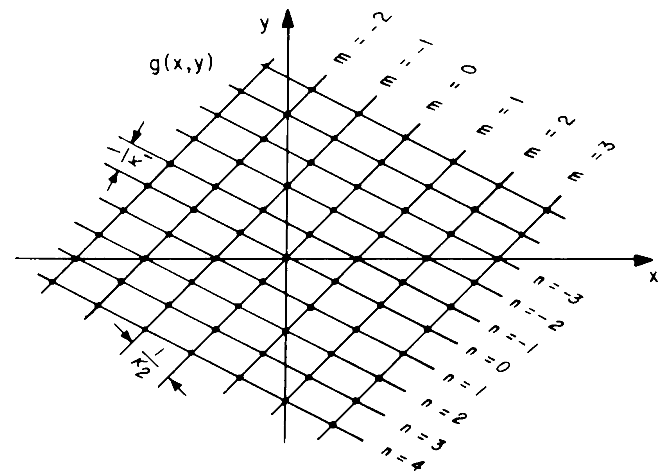
$$f(x,\ y)\mathrm{comb}(a_1 x + b_1 y + c_1,\ a_2 x + b_2 y + c_2)$$

$$g(x,y) = f(x,y)\mathrm{comb}(a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2)$$

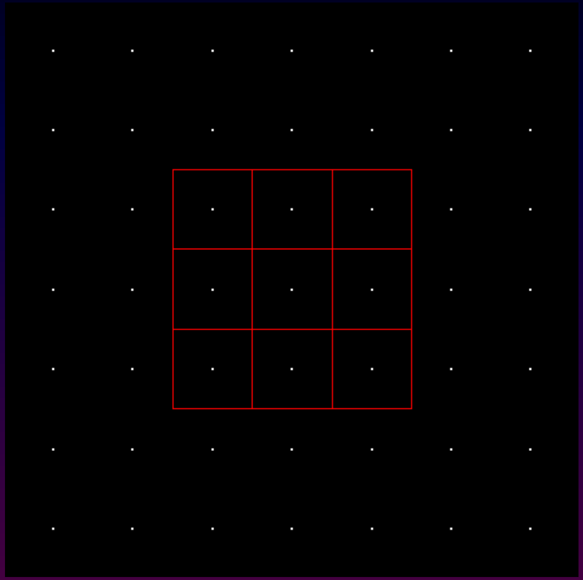
$$= \frac{1}{|D|} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f\left(x_0 + \frac{b_2 n}{D} - \frac{b_1 m}{D}, y_0 - \frac{a_2 n}{D} + \frac{a_1 m}{D}\right)$$

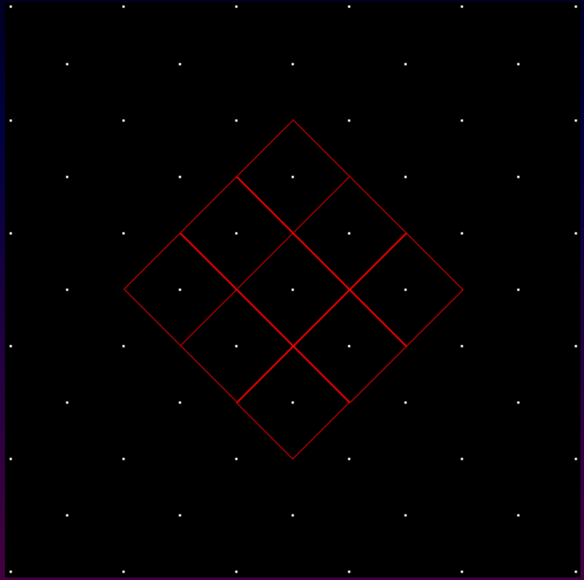
$$\times \delta\left(x - x_0 - \frac{b_2 n}{D} + \frac{b_1 m}{D}\right) \delta\left(y - y_0 + \frac{a_2 n}{D} - \frac{a_1 m}{D}\right),$$

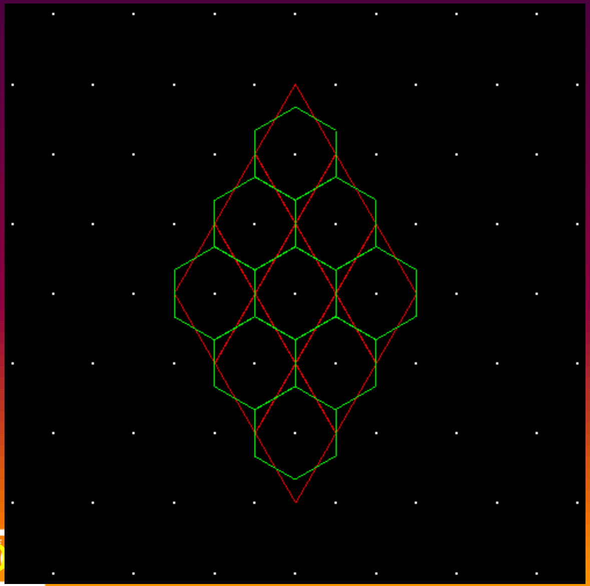
D = determinant of matrix [a_1 b_1;a_2 b_2]



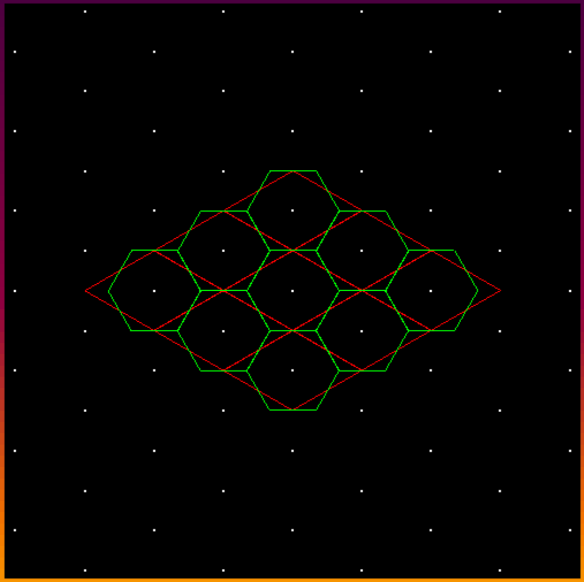
2D Sampling strategies

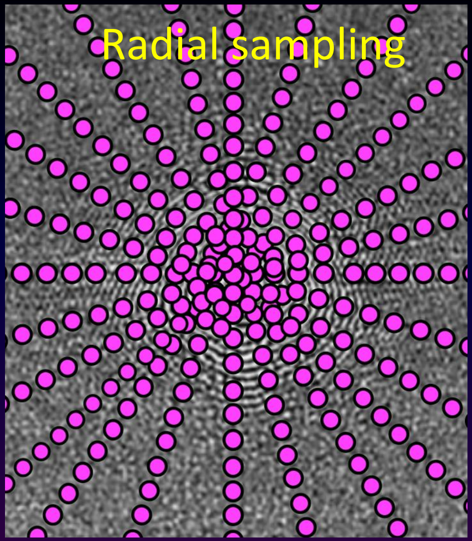




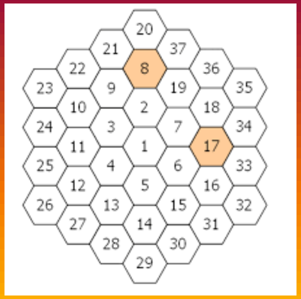


David Martindale, April 2004





Compressed sensing



Fourier Transform - properties

Definições:

$$\mathcal{F}\{g\} = \iint_{-\infty}^{\infty} g(x, y)\exp[-j2\pi(f_X x + f_Y y)]\, dx\, dy$$

$$\mathcal{F}^{-1}\{G\} = \iint_{-\infty}^{\infty} G(f_X, f_Y)\exp[j2\pi(f_X x + f_Y y)]\, df_X\, df_Y$$

Dupla transformação:

$$\mathcal{F}\mathcal{F}^{-1}\{g(x, y)\} = \mathcal{F}^{-1}\mathcal{F}\{g(x, y)\} = g(x, y)$$

Linearidade:

$$\mathcal{F}\{\alpha g + \beta h\} = \alpha\mathcal{F}\{g\} + \beta\mathcal{F}\{h\}$$

Translação:

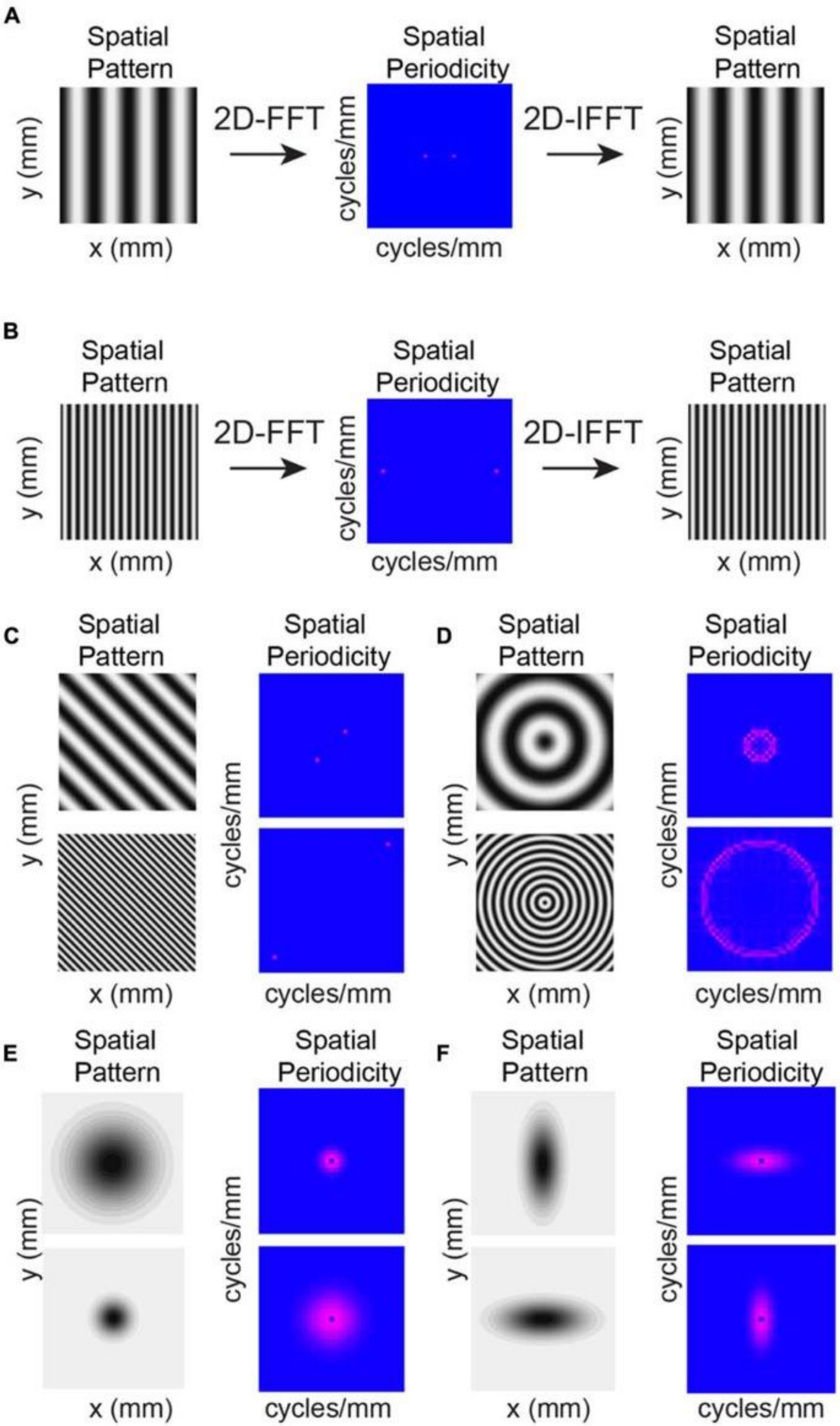
$$\mathcal{F}\{g(x - a, y - b)\} = G(f_X, f_Y)\exp[-j2\pi(f_X a + f_Y b)]$$

Escala:

$$\mathcal{F}\{g(ax, by)) = \frac{1}{|ab|} G\left(\frac{f_X}{a}, \frac{f_Y}{b}\right)$$

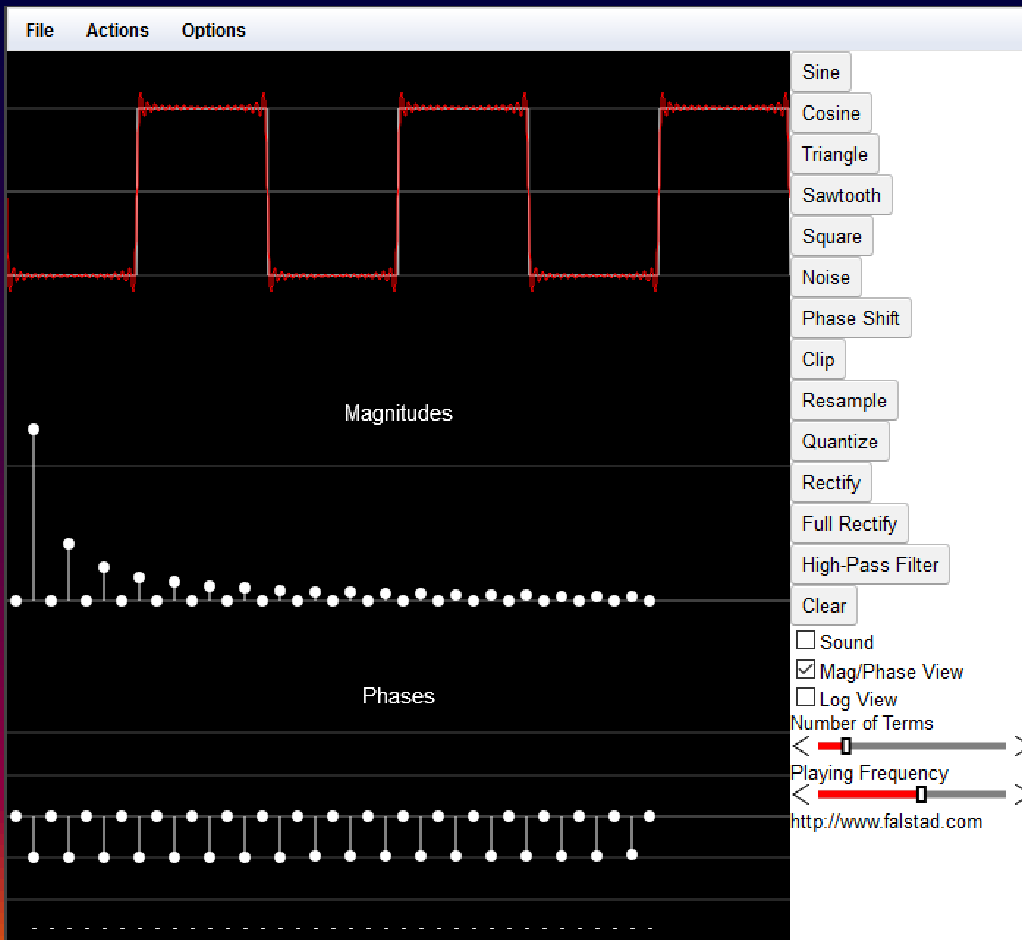
Energia / T. Parseval

$$\iint_{-\infty}^{\infty} |g(x, y)|^2\, dx\, dy = \iint_{-\infty}^{\infty} |G(f_X, f_Y)|^2\, df_X\, df_Y$$



Fourier analysis & synthesis

https://www.researchgate.net/figure/Example-2D-Fourier-Analysis-FFT-images-demonstrate-conversion-from-space-to-spatial_fig5_260999966



Fourire Analysis and Synthesis (1D)

http://www.falstad.com/fourier/

Physical meaning of 2D - TF

http://fourier.eng.hmc.edu/e161/lectures/fourier/node10.html

$$f(x,y) = \int\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(xu+yv)}\, du\, dv$$

$$e^{j2\pi(xu+yv)} = cos(2\pi(xu+yv)) + j\sin(2\pi(xu+yv))$$

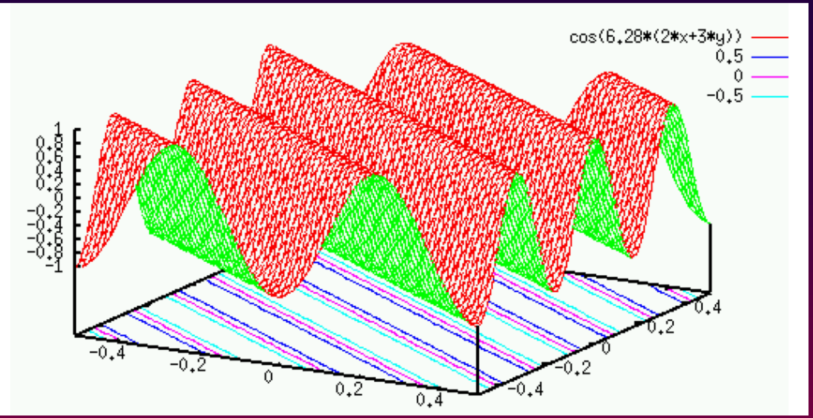
$$\begin{cases} w \triangleq \sqrt{u^2+v^2} \\ \theta \triangleq tan^{-1}[v/u] \end{cases} \quad \text{and} \quad \begin{cases} u = w\ cos\theta \\ v = w\ sin\theta \end{cases}$$

$$\mathbf{n} \triangleq [cos\theta, sin\theta]^T = \left[\frac{u}{w}, \frac{v}{w}\right]^T = \frac{1}{w}[u,\ v]^T$$

$$e^{j2\pi(xu+yv)} = e^{j2\pi w\, \mathbf{r}\cdot\mathbf{n}} = \cos(2\pi w\, \mathbf{r}\cdot\mathbf{n}) + j\ \sin(2\pi w\, \mathbf{r}\cdot\mathbf{n})$$

The function *cos* ($2\pi w$ **r.n**) represents a planar sinusoidal wave in the x-y plane, with:

- **frequency** $w = \sqrt{u^2+v^2}$
- **direction** $\underline{\mathbf{n}}$ with angle $\theta = tan^{-1}(v/u)$



$$F(u,v) = F_r(u,v) + jF_j(u,v)$$

$$\begin{cases} |F(u,v)| = \sqrt{F_r(u,v)^2 + F_j(u,v)^2} \\ \angle F(u,v) = \tan^{-1}[F_j(u,v)/F_r(u,v)] \end{cases} \quad \text{and} \quad \begin{cases} F_r(u,v) = |F(u,v)|\ \cos\angle F(u,v) \\ F_j(u,v) = |F(u,v)|\ \sin\angle F(u,v) \end{cases}$$

$$F(u,v) = F_r(u,v) + jF_j(u,v) = |F(u,v)|\,[\cos\angle F(u,v) + j\ \sin\angle F(u,v)]$$

Physical meaning of 2D - TF

http://fourier.eng.hmc.edu/e161/lectures/fourier/node10.html

For real signals, $f(x,y) = f_r(x,y) + i\, f_i(x,y)$, with $f_i(x,y)=0$:

$$f(x,y) = \int\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(xu+yv)}\, du\, dv$$

$$= \int\int_{-\infty}^{\infty} [F_r(u,v)\cos(2\pi w\, \mathbf{n}\cdot\mathbf{r}) - F_j(u,v)\sin(2\pi w\, \mathbf{n}\cdot\mathbf{r})]\ du\, dv$$

$$= \int\int_{-\infty}^{\infty} |F(u,v)|\, [\cos\angle F(u,v)\cos(2\pi w\, \mathbf{n}\cdot\mathbf{r}) - \sin\angle F(u,v)\sin(2\pi w\, \mathbf{n}\cdot\mathbf{r})]\ du\, dv$$

$$= \int\int_{-\infty}^{\infty} |F(u,v)|\ \cos[(2\pi w\, \mathbf{n}\cdot\mathbf{r}) + \angle F(u,v)]\ du\, dv$$

The FT allows representing a **real** signal *f(x,y)* as a linear superposition of an infinite number of 2D spatial sinusoids with:

- **amplitude** $|F(u,v)| = \sqrt{F_r(u,v)^2 + F_j(u,v)}$
- **phase** $\angle F(u,v) = tan^{-1}(F_j(u,v)/F_r(u,v))$
- **frequency** $w = \sqrt{u^2 + v^2}$
- **direction** $\theta = tan^{-1}(v/u)$

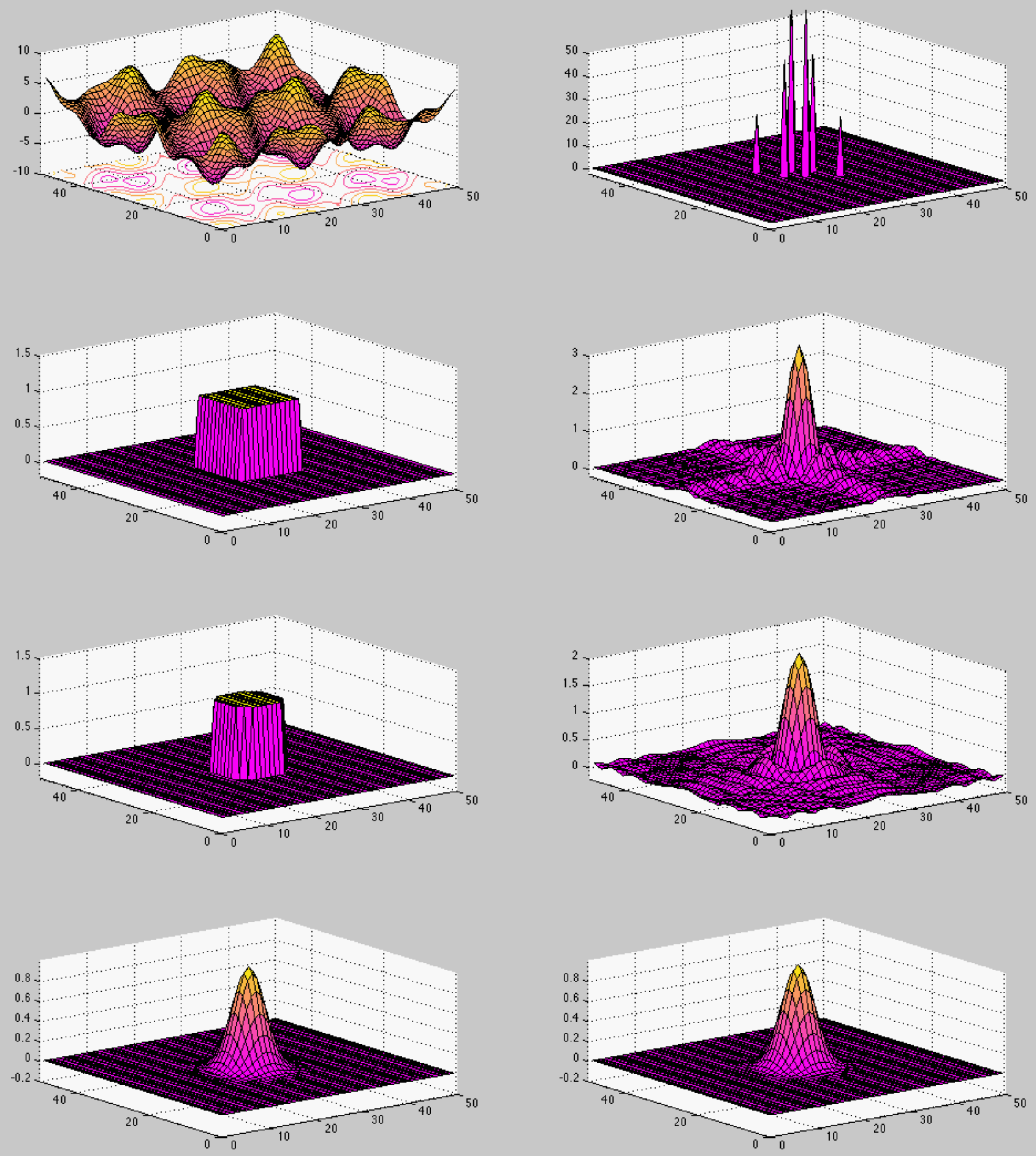
Integration variables, *u*,*v*, are spatial frequencies

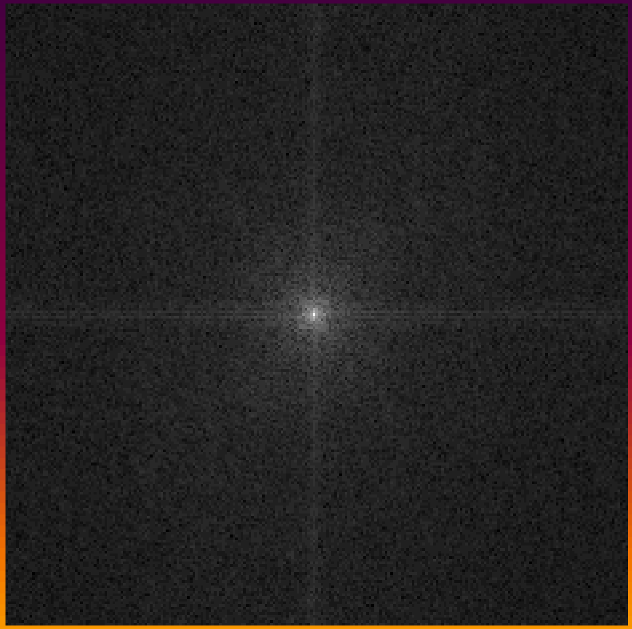
Symmetry Properties of Fourier Transforms

$f(x)$	$F(\xi)$
Complex, no symmetry	Complex, no symmetry
Hermitian	Real, no symmetry
Antihermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Real, no symmetry	Hermitian
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, no symmetry	Antihermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

Physical meaning of 2D - TF

http://fourier.eng.hmc.edu/e161/lectures/fourier/node10.html





Fourier Transform - properties

Convolução:

$$f(x,y) = g ** h = \iint_{-\infty}^{\infty} g(\xi,\eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta$$

TF da convolução:

$$F(f_x,f_y) = \mathcal{F}\left\{ \iint_{-\infty}^{\infty} g(\xi,\eta)\, h(x-\xi, y-\eta)\, d\xi\, d\eta \right\} = G(f_X, f_Y)\, H(f_X, f_Y)$$

TF de um produto:

$$\mathcal{F}\{ f(x,y) \} = \mathcal{F}\{g \times h\} = G(f_x,f_y) ** H(f_x,f_y)$$

Elemento neutro da convolução: $\delta(\xi,\eta)$

$$\delta ** g = g ** \delta = g$$

Properties of the FT

Symmetry Properties of Fourier Transforms

$f(x)$	$F(\xi)$
Complex, no symmetry	Complex, no symmetry
Hermitian	Real, no symmetry
Antihermitian	Imaginary, no symmetry
Complex, even	Complex, even
Complex, odd	Complex, odd
Real, no symmetry	Hermitian
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, no symmetry	Antihermitian
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

Properties of Fourier Transforms

A_1 and A_2 arbitrary constants; x_0 and ξ_0 real constants
b and d real nonzero constants; k a positive integer

$g(x)=\int_{-\infty}^{\infty} G(\beta)e^{j2\pi\beta x}d\beta$	$G(\xi)=\int_{-\infty}^{\infty} g(\alpha)e^{-j2\pi\alpha\xi}d\alpha$
$f(\pm x)$	$F(\pm\xi)$
$f^*(\pm x)$	$F^*(\mp\xi)$
$F(\pm x)$	$f(\mp\xi)$
$F^*(\pm x)$	$f^*(\pm\xi)$
$f\left(\frac{x}{b}\right)$	$\lvert b\rvert F(b\xi)$
$\lvert d\rvert f(dx)$	$F\left(\frac{\xi}{d}\right)$
$f(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}F(\xi)$
$e^{\pm j2\pi\xi_0 x}f(x)$	$F(\xi\mp\xi_0)$
$f\left(\frac{x\pm x_0}{b}\right)$	$\lvert b\rvert e^{\pm j2\pi x_0\xi}F(b\xi)$
$\lvert d\rvert e^{\pm j2\pi\xi_0 x}f(dx)$	$F\left(\frac{\xi\mp\xi_0}{d}\right)$
$f^{(k)}(x)$	$(j2\pi\xi)^k F(\xi)$
$(-j2\pi x)^k f(x)$	$F^{(k)}(\xi)$
$\int_{-\infty}^{x} f(\alpha)d\alpha$	$\frac{1}{j2\pi\xi}F(\xi)+\frac{F(0)}{2}\delta(\xi)$
$\frac{1}{-j2\pi x}f(x)+\frac{f(0)}{2}\delta(x)$	$\int_{-\infty}^{\xi} F(\beta)d\beta$
$h(x)$	$H(\xi)$
$A_1 f(x)+A_2 h(x)$	$A_1 F(\xi)+A_2 H(\xi)$
$f(x)*h(x)$	$F(\xi)H(\xi)$
$f(x)h(x)$	$F(\xi)*H(\xi)$
$f(x)\star h(x)$	$F(\xi)H(-\xi)$
$f(x)h(-x)$	$F(\xi)\star H(\xi)$
$\gamma_{fh}(x)=f(x)\star h^*(x)$	$F(\xi)H^*(\xi)$
$f(x)h^*(x)$	$\gamma_{FH}(\xi)=F(\xi)\star H^*(\xi)$
$\gamma_f(x)=f(x)\star f^*(x)$	$\lvert F(\xi)\rvert^2$
$\lvert f(x)\rvert^2$	$\gamma_F(\xi)=F(\xi)\star F^*(\xi)$
$\sum_{n=-\infty}^{\infty} f(x-nd)$	$\frac{1}{\lvert d\rvert}\sum_{n=-\infty}^{\infty} F\left(\frac{n}{d}\right)\delta\left(\xi-\frac{n}{d}\right)$
$\frac{1}{\lvert b\rvert}\sum_{n=-\infty}^{\infty} f\left(\frac{n}{b}\right)\delta\left(x-\frac{n}{b}\right)$	$\sum_{n=-\infty}^{\infty} F(\xi-nb)$

Wiener-Khinchin theorem: $\gamma_f(x)=f(x)\star f^*(x)$ ↔ $\lvert F(\xi)\rvert^2$

Autocorrelation is the FT of the Power Spectrum Density

Fourier Transform pairs (Gaskill)

x_0 and ξ_0 real constants k a nonnegative integer
a and c real constants x and ξ real variables

$g(x)=\int_{-\infty}^{\infty} G(\beta)e^{j2\pi x\beta}d\beta$	$G(\xi)=\int_{-\infty}^{\infty} g(\alpha)e^{-j2\pi\xi\alpha}d\alpha$
1	$\delta(\xi)$
$\delta(x)$	1
$\delta(x\pm x_0)$	$e^{\pm j2\pi x_0\xi}$
$e^{\pm j2\pi\xi_0 x}$	$\delta(\xi\mp\xi_0)$
$\cos(2\pi\xi_0 x)$	$\frac{1}{2\lvert\xi_0\rvert}\delta\delta\left(\frac{\xi}{\xi_0}\right)$
$\frac{1}{2\lvert x_0\rvert}\delta\delta\left(\frac{x}{x_0}\right)$	$\cos(2\pi x_0\xi)$
$\sin(2\pi\xi_0 x)$	$\frac{j}{2\lvert\xi_0\rvert}\delta\delta\left(\frac{\xi}{\xi_0}\right)$
$\frac{j}{2\lvert x_0\rvert}\delta\delta\left(\frac{x}{x_0}\right)$	$-\sin(2\pi x_0\xi)$
$\mathrm{rect}(x)$	$\mathrm{sinc}(\xi)$
$\mathrm{sinc}(x)$	$\mathrm{rect}(\xi)$
$\mathrm{tri}(x)$	$\mathrm{sinc}^2(\xi)$
$\mathrm{sinc}^2(x)$	$\mathrm{tri}(\xi)$
$\mathrm{sgn}(x)$	$\frac{1}{j\pi\xi}$
$\frac{1}{j\pi x}$	$-\mathrm{sgn}(\xi)$
$\mathrm{step}(x)$	$\frac{1}{2}\delta(\xi)+\frac{1}{j2\pi\xi}$
$\frac{1}{2}\delta(x)-\frac{1}{j2\pi x}$	$\mathrm{step}(\xi)$
$\mathrm{ramp}(x)$	$\frac{1}{4\pi^2}\left[j\pi\delta^{(1)}(\xi)-\frac{1}{\xi^2}\right]$
$\frac{1}{4\pi^2}\left[\frac{1}{x^2}+j\pi\delta^{(1)}(x)\right]$	$-\mathrm{ramp}(\xi)$
$e^{-\lvert x\rvert}$	$\frac{2}{1+(2\pi\xi)^2}$
$\frac{2}{1+(2\pi x)^2}$	$e^{-\lvert\xi\rvert}$
$e^{-x}\mathrm{step}(x)$	$\frac{1}{1+j2\pi\xi}$
$\frac{1}{1-j2\pi x}$	$e^{-\xi}\mathrm{step}(\xi)$
x^k	$\left(\frac{-1}{j2\pi}\right)^k\delta^{(k)}(\xi)$
$\left(\frac{1}{j2\pi}\right)^k\delta^{(k)}(x)$	ξ^k
$\mathrm{comb}(x)$	$\mathrm{comb}(\xi)$
$\mathrm{Gaus}(x)$	$\mathrm{Gaus}(\xi)$
$\mathrm{sech}(\pi x)$	$\mathrm{sech}(\pi\xi)$
$\frac{1}{\lvert x\rvert^{1/2}}$	$\frac{1}{\lvert\xi\rvert^{1/2}}$
$\cos\pi(x^2-\frac{1}{8})$	$\cos\pi(\xi^2-\frac{1}{8})$
$\sin\pi(x^2-\frac{1}{8})$	$-\sin\pi(\xi^2-\frac{1}{8})$
$\exp[\pm j\pi(x^2-\frac{1}{8})]$	$\exp[\mp j\pi(\xi^2-\frac{1}{8})]$
$\cos(\pi x^2)$	$\cos\pi(\xi^2-\frac{1}{4})$
$\sin(\pi x^2)$	$-\sin\pi(\xi^2-\frac{1}{4})$
$\exp(\pm j\pi x^2)$	$\exp\left(\pm j\frac{\pi}{4}\right)\exp(\mp j\pi\xi^2)$
$\exp\left[-\pi\left(\frac{x^2}{a+jc}\right)\right],\ a\geqslant 0,\ a^2+c^2<\infty$	$(a+jc)^{1/2}\exp[-\pi(a+jc)\xi^2]$

FT of 2D functions (rectangular coordinates; Goodman)

Function	Transform
$\exp[-\pi(a^2x^2 + b^2y^2)]$	$\frac{1}{\lvert ab \rvert} \exp\left[-\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\mathrm{rect}(ax)\,\mathrm{rect}(by)$	$\frac{1}{\lvert ab \rvert} \mathrm{sinc}(f_X/a)\,\mathrm{sinc}(f_Y/b)$
$\Lambda(ax)\,\Lambda(by)$	$\frac{1}{\lvert ab \rvert} \mathrm{sinc}^2(f_X/a)\,\mathrm{sinc}^2(f_Y/b)$
$\delta(ax, by)$	$\frac{1}{\lvert ab \rvert}$
$\exp[j\pi(ax + by)]$	$\delta(f_X - a/2, f_Y - b/2)$
$\mathrm{sgn}(ax)\,\mathrm{sgn}(by)$	$\frac{ab}{\lvert ab \rvert} \frac{1}{j\pi f_X} \frac{1}{j\pi f_Y}$
$\mathrm{comb}(ax)\,\mathrm{comb}(by)$	$\frac{1}{\lvert ab \rvert}\mathrm{comb}(f_X/a)\,\mathrm{comb}(f_Y/b)$
$\exp[j\pi(a^2x^2 + b^2y^2)]$	$\frac{j}{\lvert ab \rvert} \exp\left[-j\pi\left(\frac{f_X^2}{a^2} + \frac{f_Y^2}{b^2}\right)\right]$
$\exp[-(a\lvert x \rvert + b\lvert y \rvert)]$	$\frac{1}{\lvert ab \rvert} \frac{2}{1 + (2\pi f_X/a)^2} \frac{2}{1 + (2\pi f_Y/b)^2}$

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FT of radially symmetric 2D functions (Fourier-Bessel)

$$r = \sqrt{x^2 + y^2} \qquad x = r\cos\theta$$

$$\theta = \arctan\left(\frac{y}{x}\right) \qquad y = r\sin\theta$$

$$\rho = \sqrt{f_X^2 + f_Y^2} \qquad f_X = \rho\cos\phi$$

$$\phi = \arctan\left(\frac{f_Y}{f_X}\right) \qquad f_Y = \rho\sin\phi.$$

$$g(r,\theta) = g_R(r)$$

$$G(f_X, f_Y) = \iint_{-\infty}^{\infty} g(x,y)\exp[-j2\pi(f_X x + f_Y y)]\,dx\,dy.$$

$$G_o(\rho,\phi) = G_o(\rho) = 2\pi\int_0^{\infty} r\,g_R(r)\,J_0(2\pi r\rho)\,dr.$$

$$\mathrm{circ}(r) = \begin{cases} 1 & r < 1 \\ \frac{1}{2} & r = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{B}\{\mathrm{circ}(r)\} = 2\pi\int_0^1 r\,J_0(2\pi r\rho)\,dr.$$

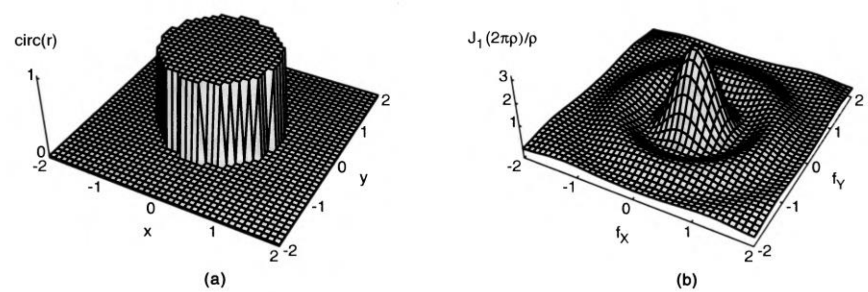


Figure 2.3 The circle function and its transform.

Convolution

Commutative Property

f(x) * h(x) = h(x) * f(x)

Distributive Property

$$[av(x)+bw(x)]*h(x)=a[v(x)*h(x)]+b[w(x)*h(x)]$$

Shift Invariance

f(x) * h(x) = g(x) → $f(x-x_0)$ * h(x) = $g(x-x_0)$

Scaling Property

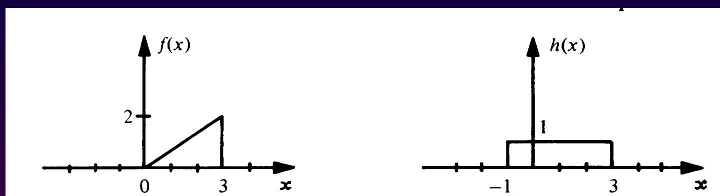
$$f\left(\frac{x}{b}\right)*h\left(\frac{x}{b}\right)=|b|\,g\left(\frac{x}{b}\right)$$

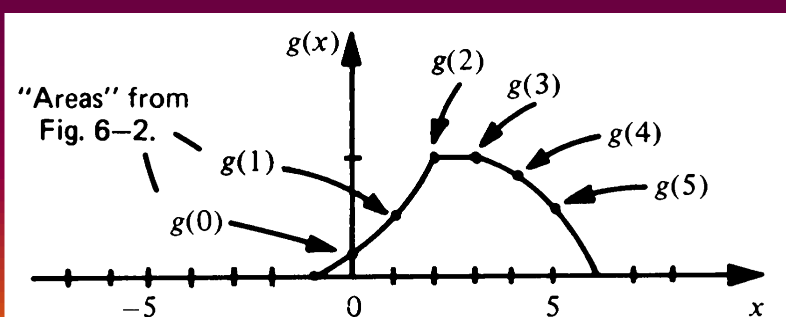
$$f(x)*\delta(x)=f(x).$$

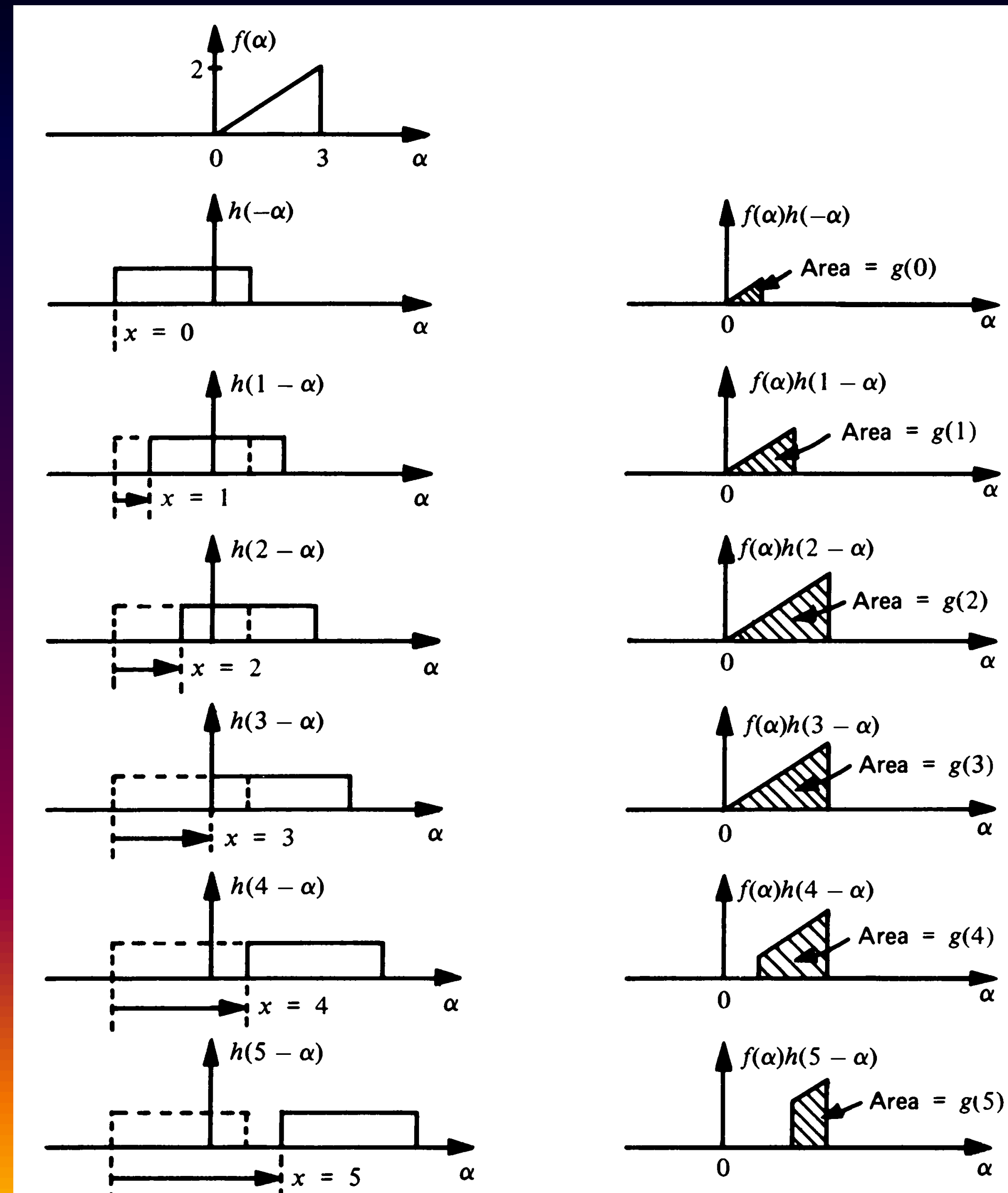
$$f(x)*\delta^{(k)}(x)=f^{(k)}(x)$$

Convolution

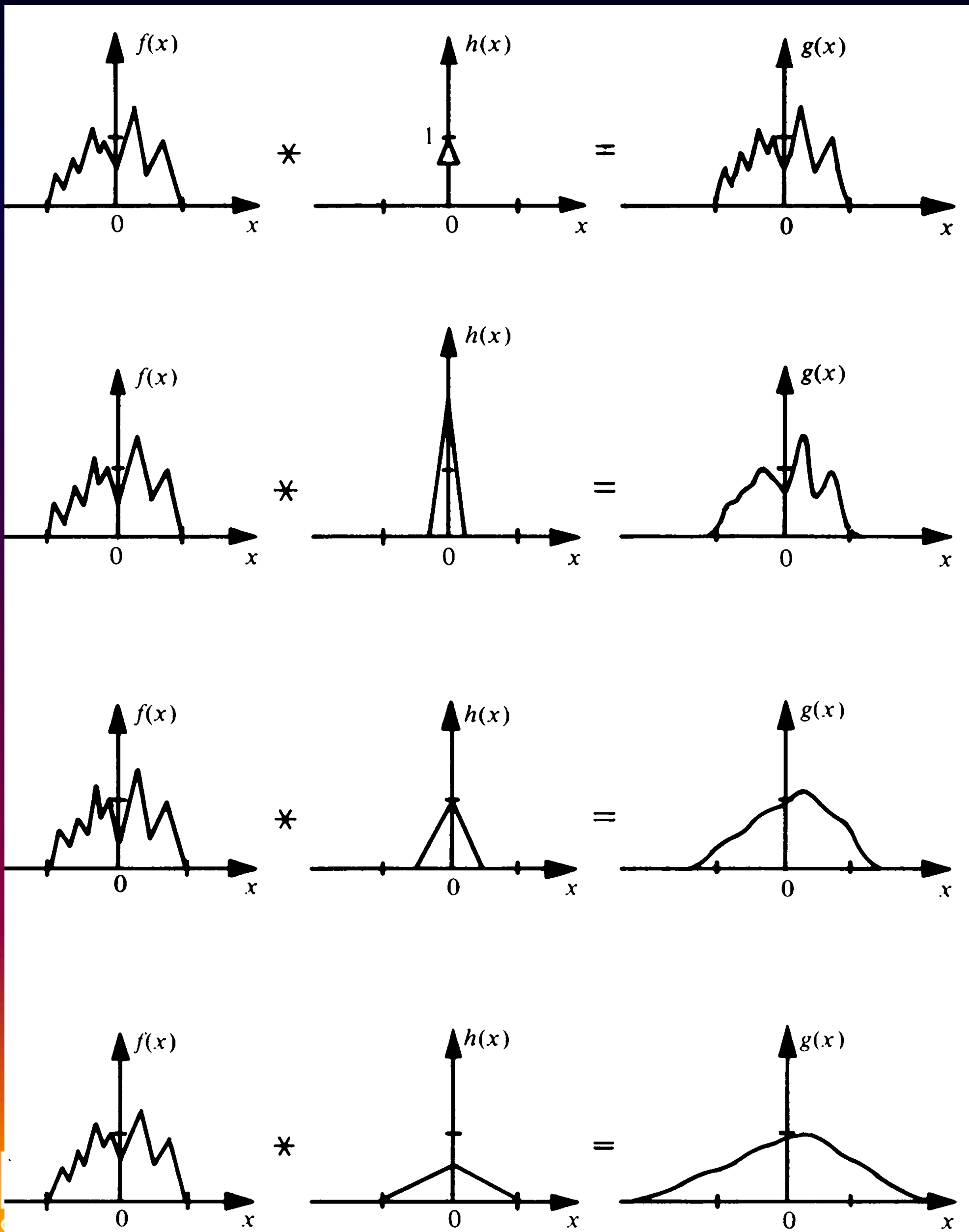
$$f(x) * h(x) = \int_{-\infty}^{\infty} f(\alpha)h(x-\alpha)d\alpha$$







Convolution – Smoothing effects



Exceptions

$$\mathrm{sinc}(x) * \mathrm{sinc}(x) = \mathrm{sinc}(x)$$

Let f(x) be a band-limited function – its Fourier transform F(ξ) has compact support (Δξ=W). If W < |b^{-1}|:

$$f(x) * \frac{1}{|b|}\mathrm{sinc}\left(\frac{x}{b}\right) = f(x)$$

Repeated Convolution

$$g(x) = f_1(x) * f_2(x) * \ldots * f_n(x).$$

It can be shown that, if functions $f_i(x)$ satisfy certain conditions, then as n → ∞, g(x) tends toward a Gaussian function.

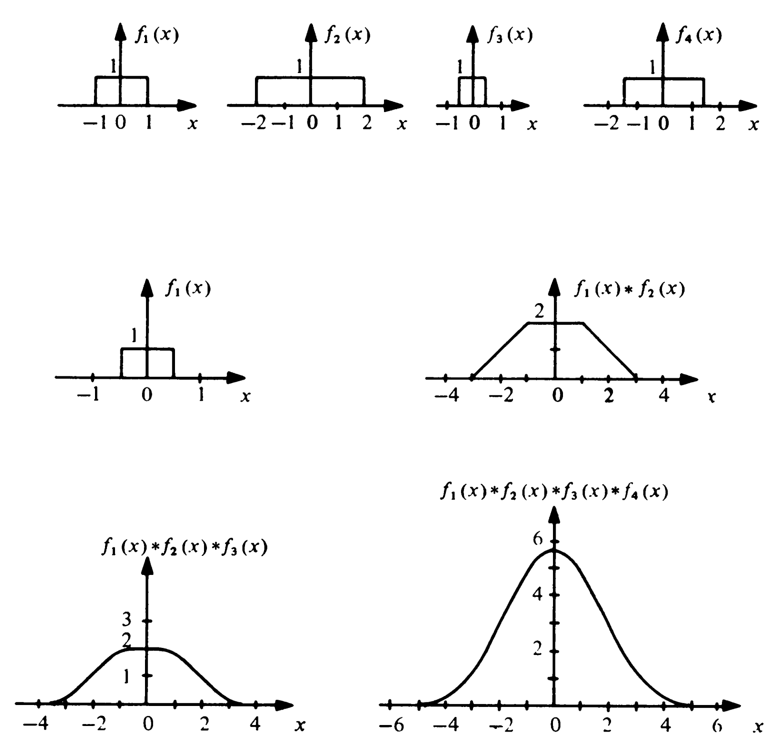


Figure 6-7 Repeated convolution of four rectangle functions.

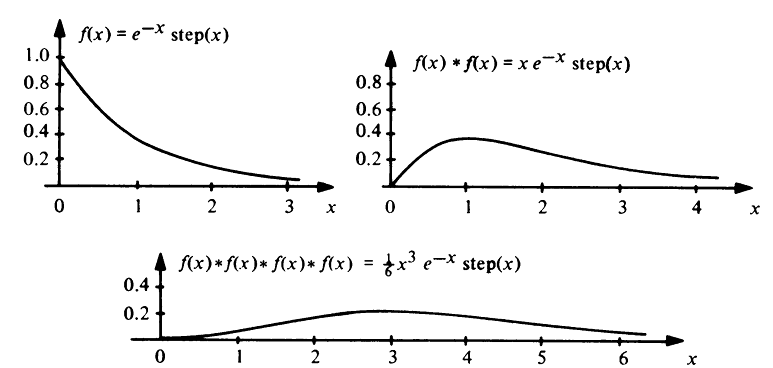


Figure 6-8 Repeated convolution of the function $\exp\{-x\}\text{step}(x)$.

1D rectangular diffraction grating

Rede

$$t(\xi) = \left[\text{rect}\frac{\xi}{a} * \sum_{n=-\infty}^{\infty} \delta(\xi - nb)\right] \text{rect}\frac{\xi}{c} = \left[\text{rect}\frac{\xi}{a} * \frac{1}{b}\text{comb}\frac{\xi}{b}\right] \text{rect}\frac{\xi}{c}$$

Transformada de Fourier

$$f(x)\delta(x-y) = f(y)(x-y)$$

$$T(f_X) = [\text{a sinc}(af_X) \ \ \text{comb}(bf_X)] * c\ \text{sinc}(cf_X)$$

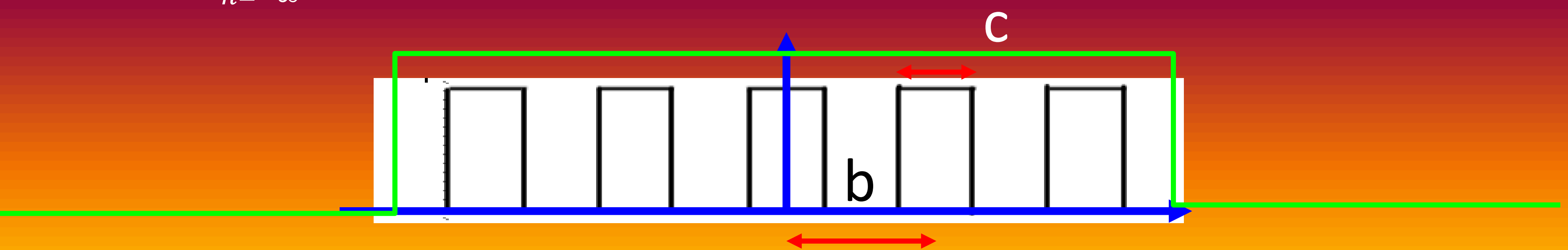
$$T(f_X) = \left[\text{a sinc}(af_X) \ \ \frac{1}{b}\sum_{n=-\infty}^{\infty} \delta\left(f_X - \frac{n}{b}\right)\right] * c\ \text{sinc}(cf_X)$$

$$\delta\left(\frac{x-y}{b}\right) = b\delta(x-y)$$

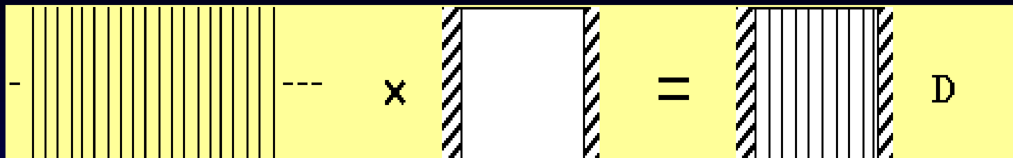
$$T(f_X) = \left[\text{a} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{a}{b}n\right) \delta\left(f_X - \frac{n}{b}\right)\right] * c\ \text{sinc}(cf_X)$$

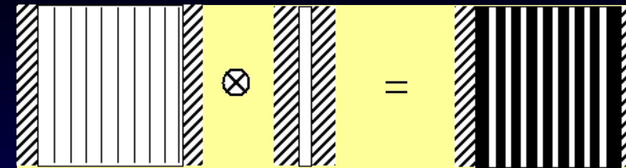
$$T(f_X) = \text{a} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{a}{b}n\right) \delta\left(f_X - \frac{n}{b}\right) * c\ \text{sinc}(cf_X)$$

$$T(f_X) = \text{ac} \sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{a}{b}n\right) \text{sinc}\left[c\left(f_X - \frac{n}{b}\right)\right]$$

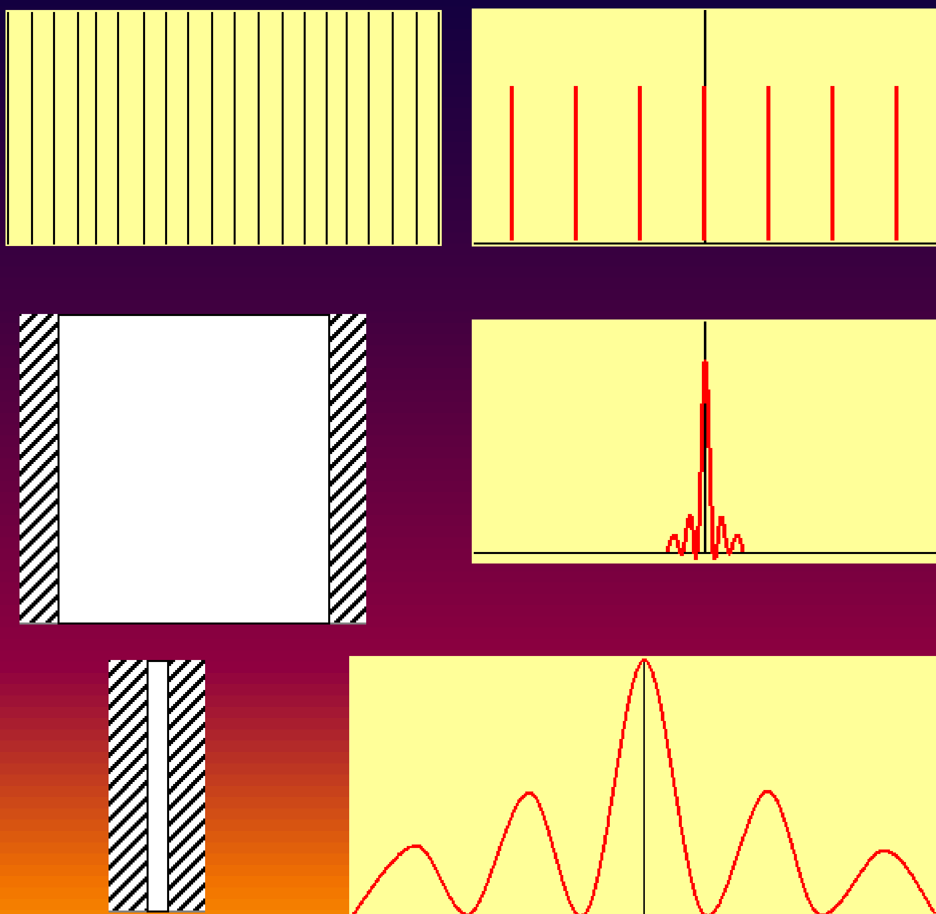


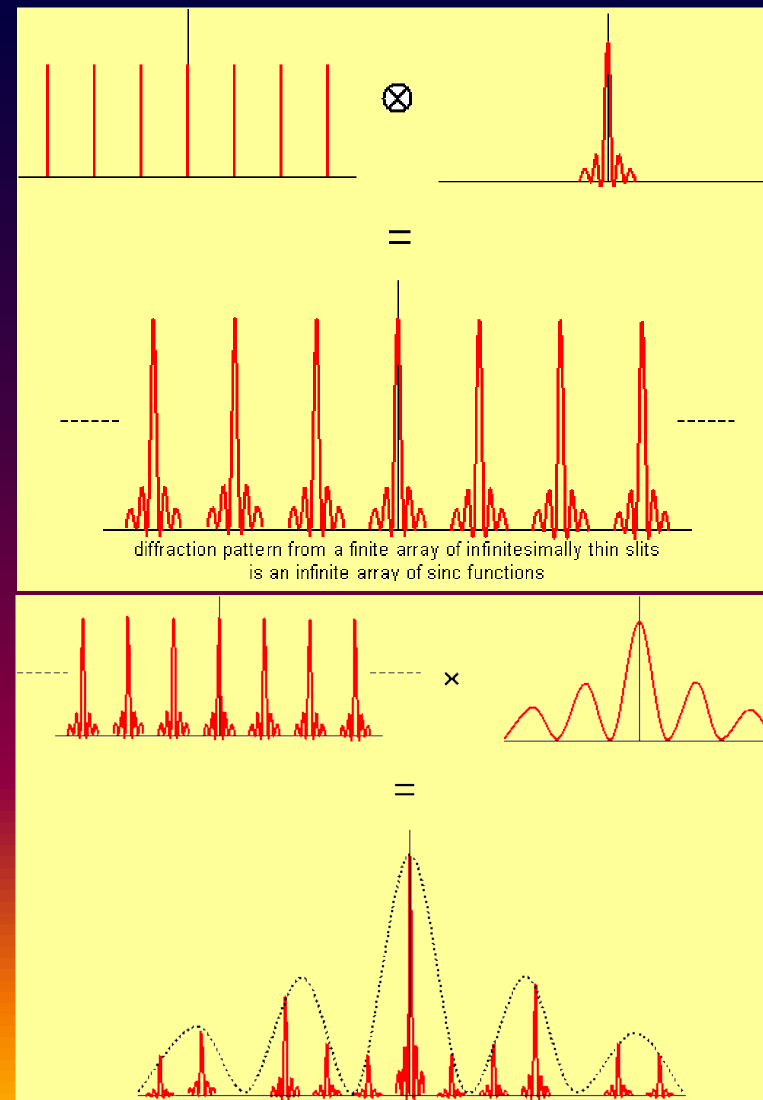
2D rectangular diffraction grating



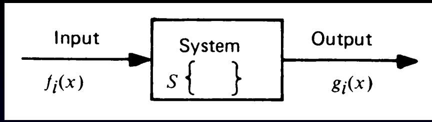


OBJECTS　　DIFFRACTION PATTERNS





https://www.doitpoms.ac.uk/tlplib/diffraction/convolution.php



Linear Systems (Goodman)

Input / Output relation

$$g_2(x_2, y_2) = \mathcal{S}\{g_1(x_1, y_1)\}$$

Linear system

$$\mathcal{S}\{ap(x_1, y_1) + bq(x_1, y_1)\} = a\mathcal{S}\{p(x_1, y_1)\} + b\mathcal{S}\{q(x_1, y_1)\}$$

Input representation

$$g_1(x_1, y_1) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta$$

System in action

$$g_2(x_2, y_2) = \mathcal{S}\left\{ \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \delta(x_1 - \xi, y_1 - \eta)\, d\xi\, d\eta \right\}$$

Output

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}\, d\xi\, d\eta$$

Impulse function

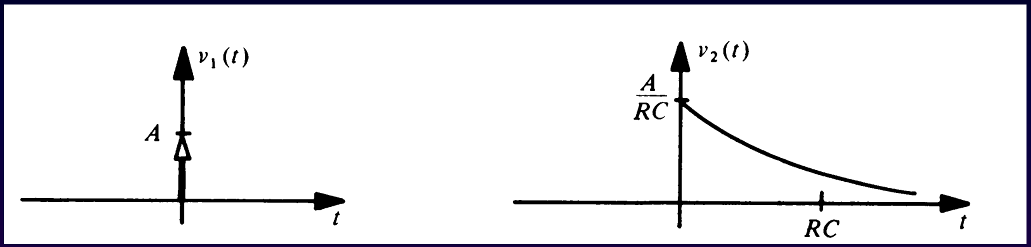
$$h(x_2, y_2; \xi, \eta) = \mathcal{S}\{\delta(x_1 - \xi, y_1 - \eta)\}$$

Output representation / superposition integral

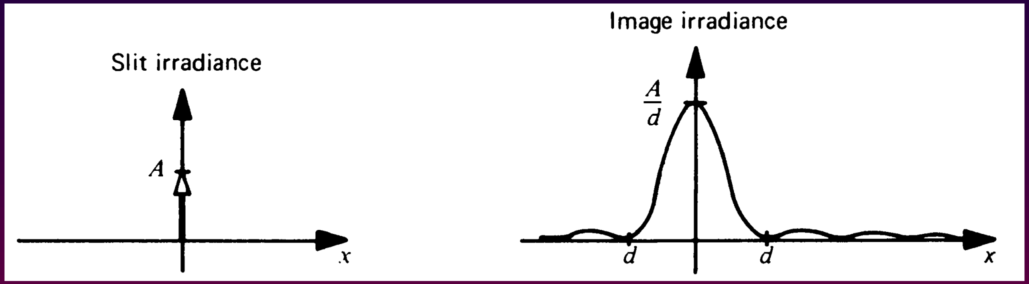
$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2, y_2; \xi, \eta)\, d\xi\, d\eta$$

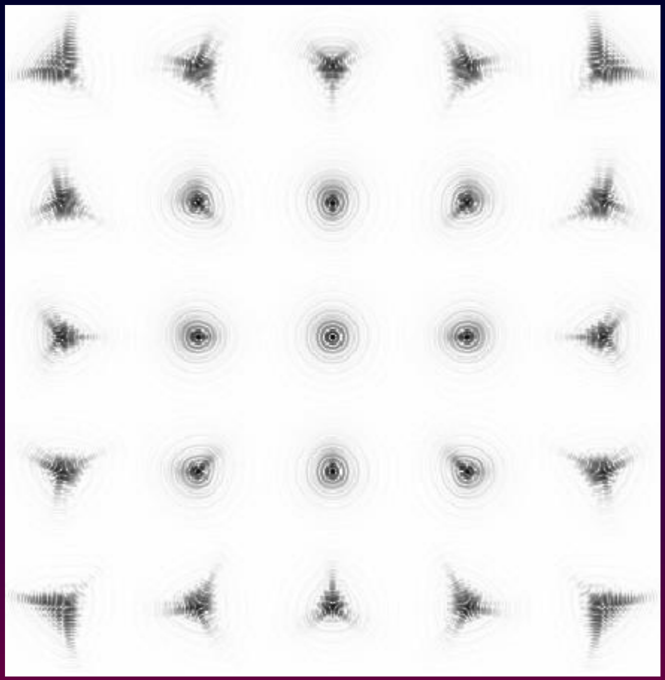
Invariant / Variant Systems

Causal (1D, t)

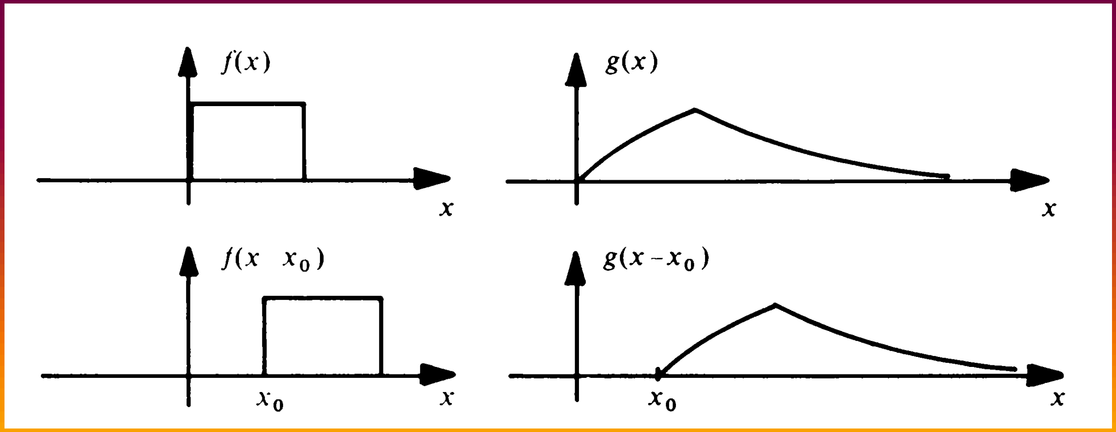


Noncausal (2D, **r**)





Space variant / anisoplanatic (2D)



Invariant / isoplanatic (1D)

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Variant / non-isoplanatic example

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Variant / non-isoplanatic example

Invariant Linear Systems

The Impulse function has the **same form** everywhere

$$h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y_2 - \eta)$$

Output as a **Convolution**

$$g_2(x_2, y_2) = \iint_{-\infty}^{\infty} g_1(\xi, \eta)\, h(x_2 - \xi, y_2 - \eta)\, d\xi\, d\eta$$

$$g_2 = g_1 \otimes h$$

Output spatial frequency **spectrum** (Fourier domain)

$$G_2(f_X, f_Y) = H(f_X, f_Y)\, G_1(f_X, f_Y)$$

Transfer function

$$H(f_X, f_Y) = \iint_{-\infty}^{\infty} h(\xi, \eta) \exp[-j2\pi(f_X \xi + f_Y \eta)]\, d\xi\, d\eta.$$

(FT of the Impulse function)

Is my System Linear Invariant?

Are you able to **describe** your system in terms of a pair of input / output **physical functions** which enable linearity and isoplanatism?

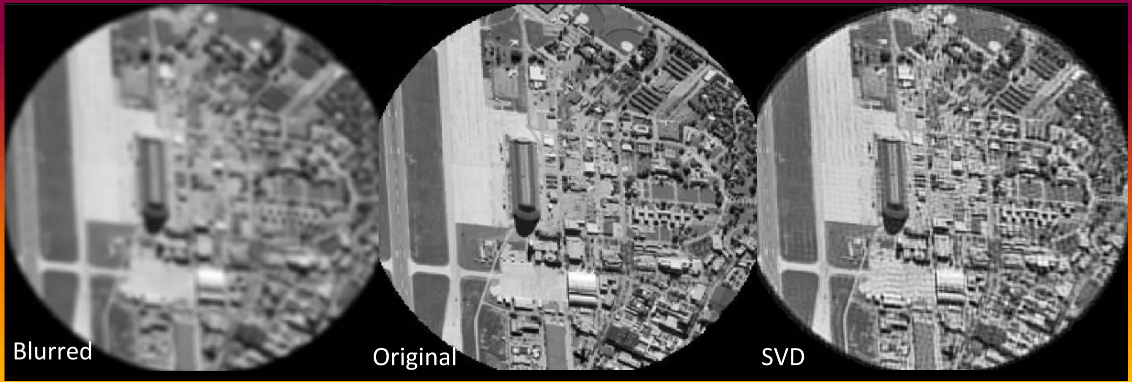
Use physics to derive the **Impulse function** and/or the **Transfer function** – or measure those functions…

Use System Theory to **compute output** from input

Are you able to describe your systems in tems of a pair of input / output functions which enable "***reasonable"*** linearity and isoplanatism within **spatial patches** (*although variable from patch to patch*)?

Can you reasonably model / parameterize the Impulse and / or the Transfer function within each patch?

Go … You will need to perform **Image Restauration** (a lot of algebra …)



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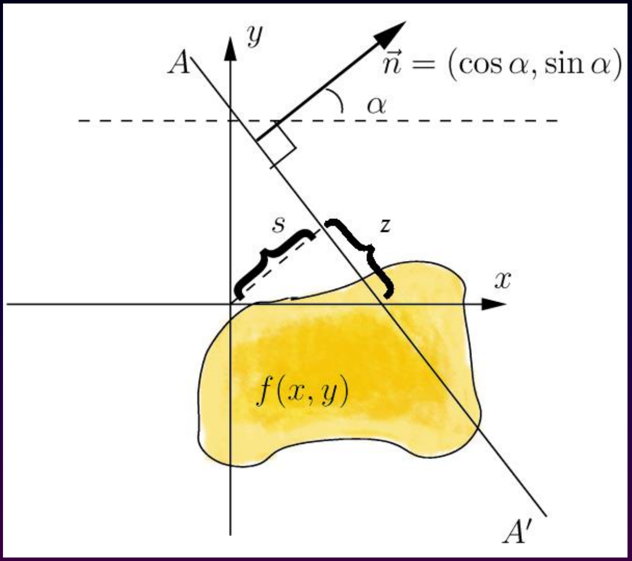
Radon transform

$$Rf(L) = \int_L f(\mathbf{x})\,|d\mathbf{x}|.$$



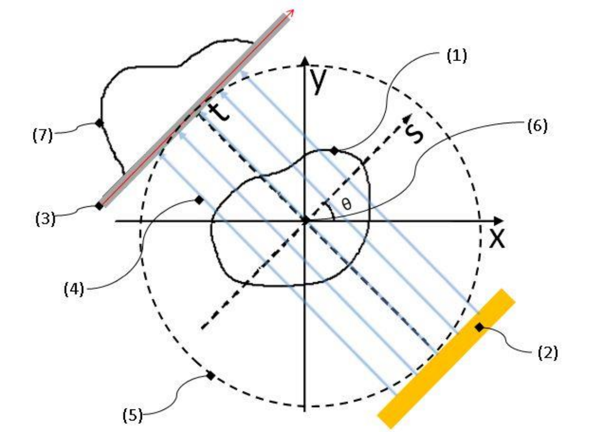
L é uma recta à distância s de O, parametrizada pelo comprimento de arco, z

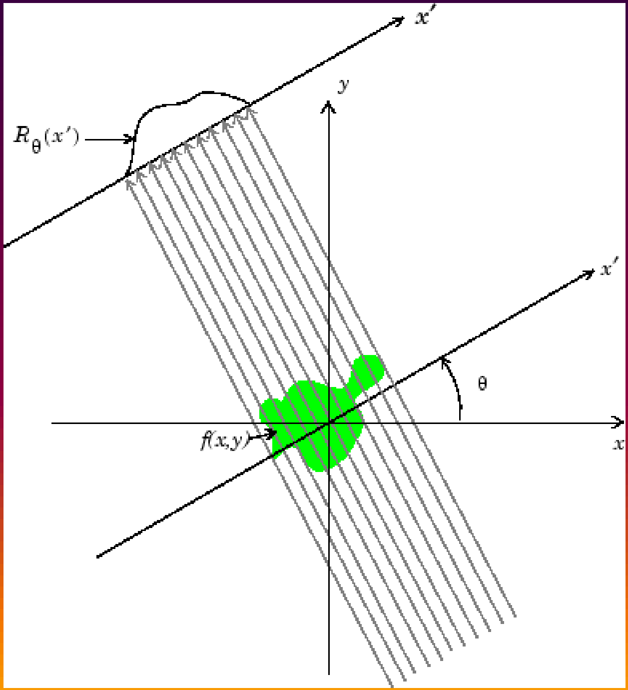
$$(x(z), y(z)) = \Big((z\sin\alpha + s\cos\alpha), (-z\cos\alpha + s\sin\alpha)\Big)$$

$$\begin{aligned} Rf(\alpha, s) &= \int_{-\infty}^{\infty} f(x(z), y(z))\,dz \\ &= \int_{-\infty}^{\infty} f\big((z\sin\alpha + s\cos\alpha), (-z\cos\alpha + s\sin\alpha)\big)\,dz \end{aligned}$$

No espaço de Radon, as variáveis independentes são α, s:

$f(x,y) \rightarrow R_f(\alpha,s)$





https://pt.wikipedia.org/wiki/Transformada_de_Radon

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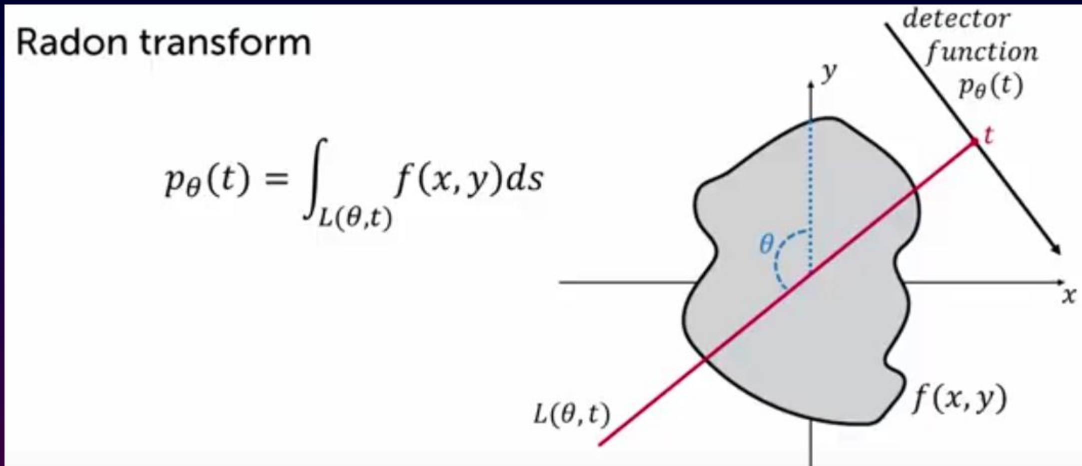
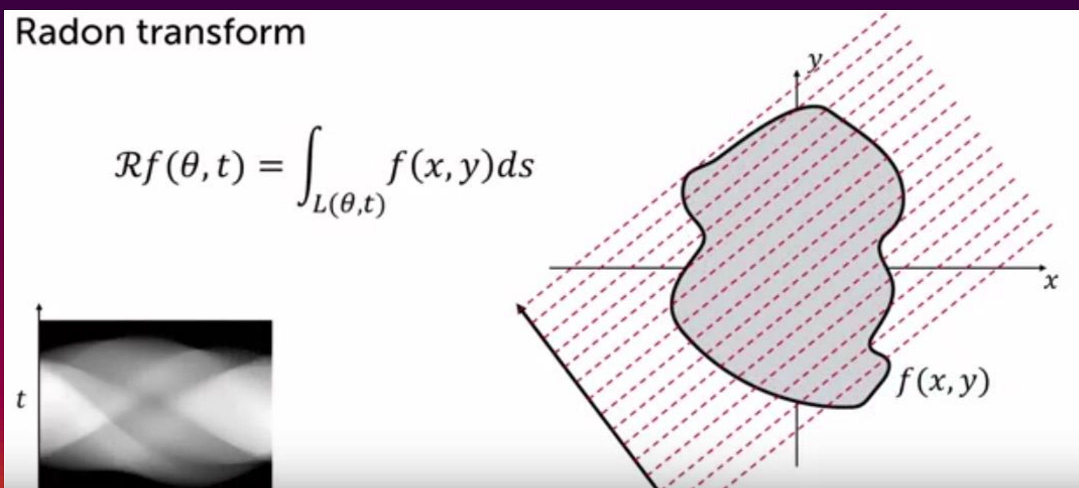
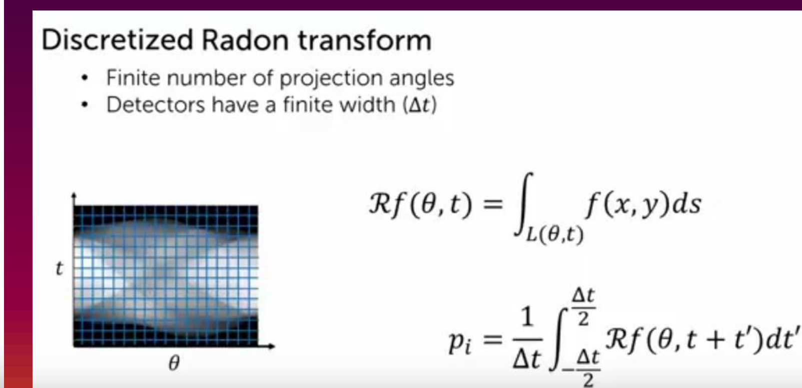
Radon transform







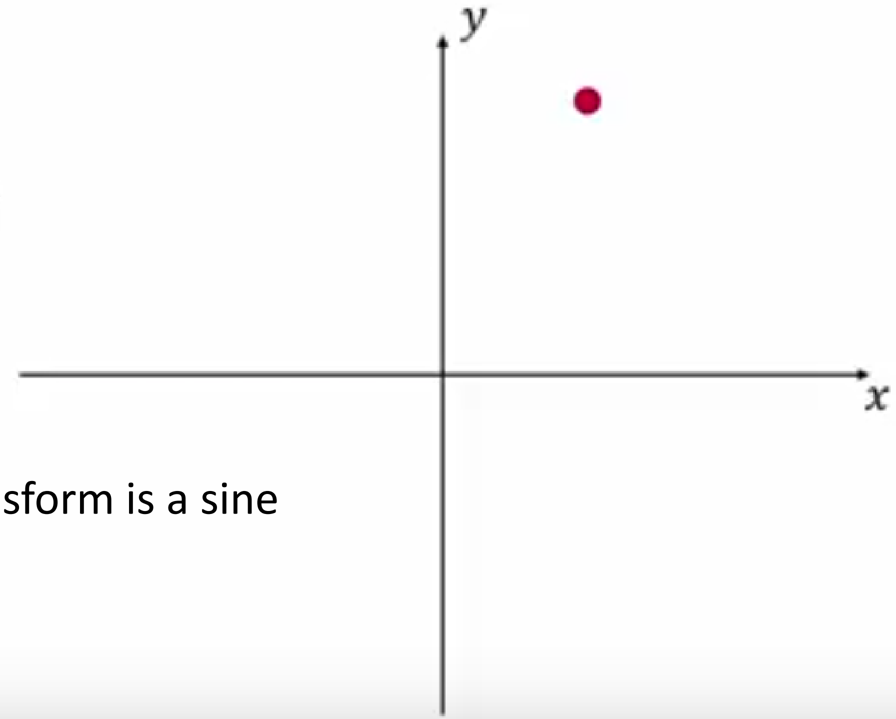
https://pt.wikipedia.org/wiki/Transformada_de_Radon
"Computed Tomography and the ASTRA Toolbox" course. University of Antwerp. September 10, 2015.
http://visielab.uantwerpen.be/astra-training

Ciências ULisboa

Radon transform

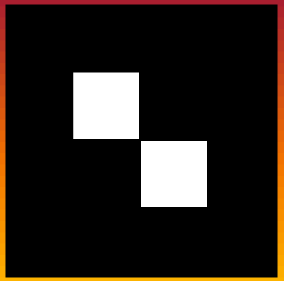
Radon transform

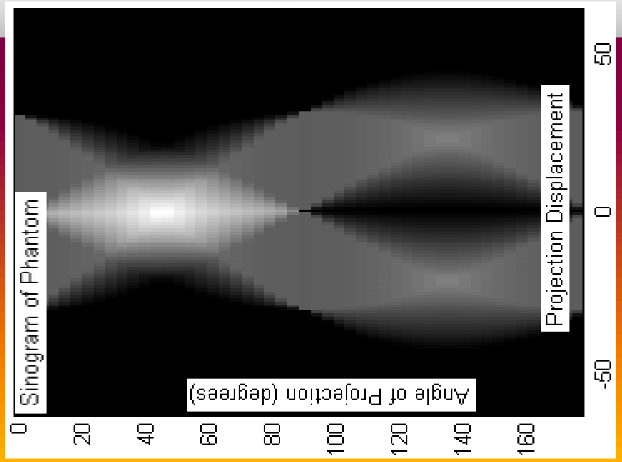
$$\boxed{\mathcal{R}f(\theta,t)} = \int_{L(\theta,t)} f(x,y)\,ds$$



Sinogram

For a δ pulse, the transform is a sine





Radon transform

Tabela 1 - Transformadas de Radon bidimensionais de algumas funções f(x, y)[19]

$f(x)$	$\phi(\rho,\theta)$
$e^{-(x^2+y^2)}$	$\sqrt{\pi}\,e^{-\rho^2}$
$(x^2+y^2)\cdot e^{-(x^2+y^2)}$	$\sqrt{\pi}\left(\rho^2+\frac{1}{2}\right)\cdot e^{-\rho^2}$
$x\cdot e^{-(x^2+y^2)}$	$\sqrt{\pi}\,e^{-\rho^2}\cdot\cos(\theta)$
$y\cdot e^{-(x^2+y^2)}$	$\sqrt{\pi}\,e^{-\rho^2}\cdot\sin(\theta)$
$x^2\cdot e^{-(x^2+y^2)}$	$\sqrt{\pi}\left(\rho^2\cos^2(\theta)+\frac{\sin^2(\theta)}{2}\right)\cdot e^{-\rho^2}$
$y^2\cdot e^{-(x^2+y^2)}$	$\sqrt{\pi}\left(\rho^2\sin^2(\theta)+\frac{\cos^2(\theta)}{2}\right)\cdot e^{-\rho^2}$
$e^{-(ax^2+by^2)}$	$\sqrt{\frac{\pi}{\lvert ab\rvert\psi(\theta)}}\cdot e^{-\frac{\rho^2}{\psi(\theta)}}$
$\delta(x-a)\cdot\delta(y-a)$	$\delta(\rho-a\cos(\theta)-b\sin(\theta))$
$\chi\left(\sqrt{x^2+y^2}\right)$	$2\sqrt{(1-\rho^2)}\cdot\chi(\rho)$
$x^2\cdot\chi\left(\sqrt{x^2+y^2}\right)$	$\sqrt{(1-\rho^2)}\cdot\left[2\rho^2\cdot\cos^2(\theta)+\frac{2}{3}(1-\rho^2)\cdot\sin^2(\theta)\right]$
$y^2\cdot\chi\left(\sqrt{x^2+y^2}\right)$	$\sqrt{(1-\rho^2)}\cdot\left[2\rho^2\cdot\sin^2(\theta)+\frac{2}{3}(1-\rho^2)\cdot\cos^2(\theta)\right]$
$(x^2+y^2)\cdot\chi\left(\sqrt{x^2+y^2}\right)$	$\frac{2}{3}\sqrt{(1-\rho^2)}\cdot(2\rho^2+1)$

- $\chi(x)$ é a função indicadora para o círculo de raio unitário[n
- $\delta(x)$ é a função impulso unitário
- $\psi(\theta)=\frac{\cos^2(\theta)}{a}+\frac{\sin^2(\theta)}{b}$
- $a,b\in\mathcal{R}$

Projection-slice theorem

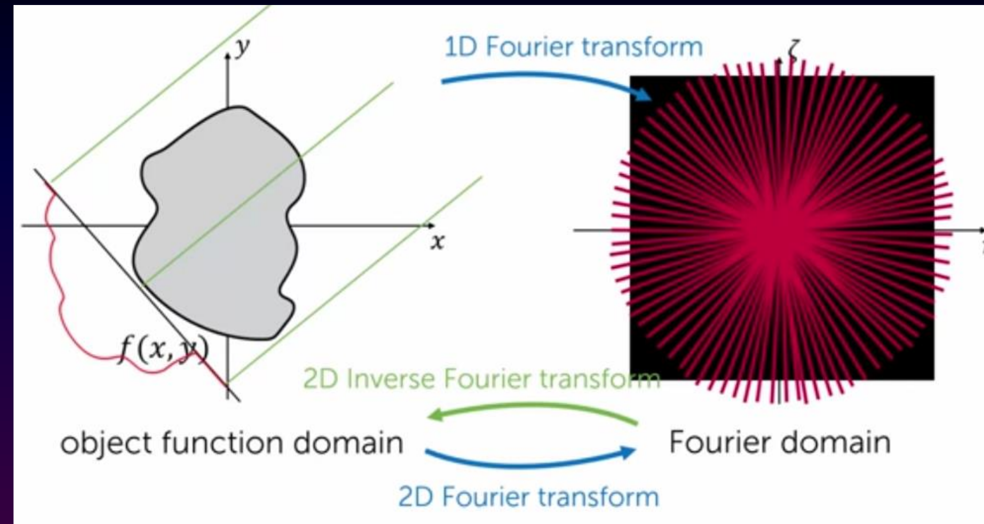
$$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy.$$

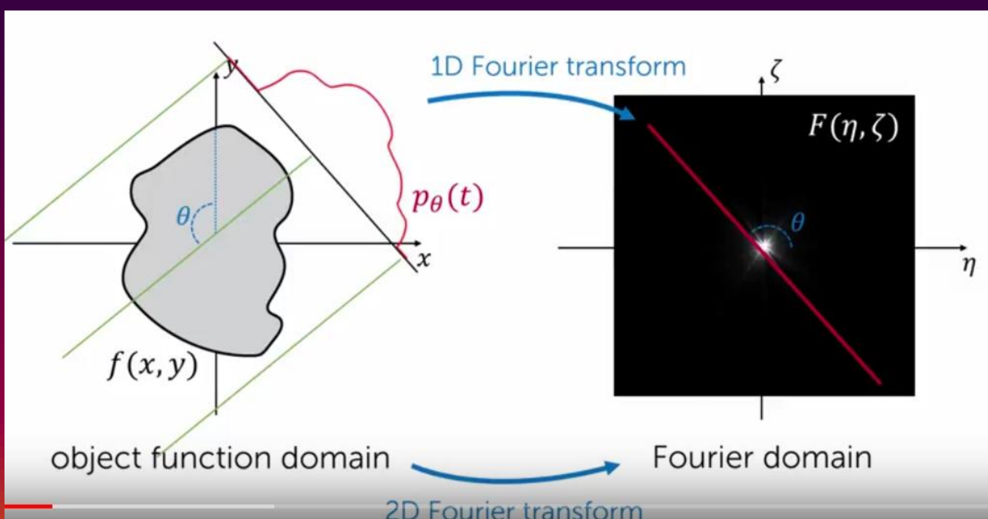
The Fourier transform of $f(x,y)$ is

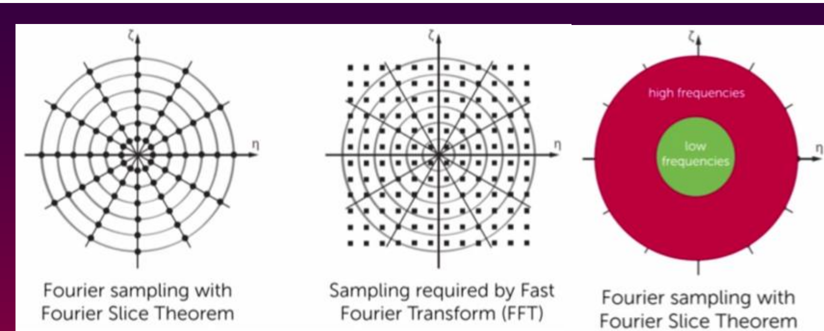
$$F(k_x, k_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i(xk_x + yk_y)}\, dxdy.$$

The slice is then $s(k_x)$

$$s(k_x) = F(k_x, 0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-2\pi i x k_x}\, dxdy$$
$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x,y)\, dy\right] e^{-2\pi i x k_x}\, dx$$
$$= \int_{-\infty}^{\infty} p(x)\, e^{-2\pi i x k_x}\, dx$$





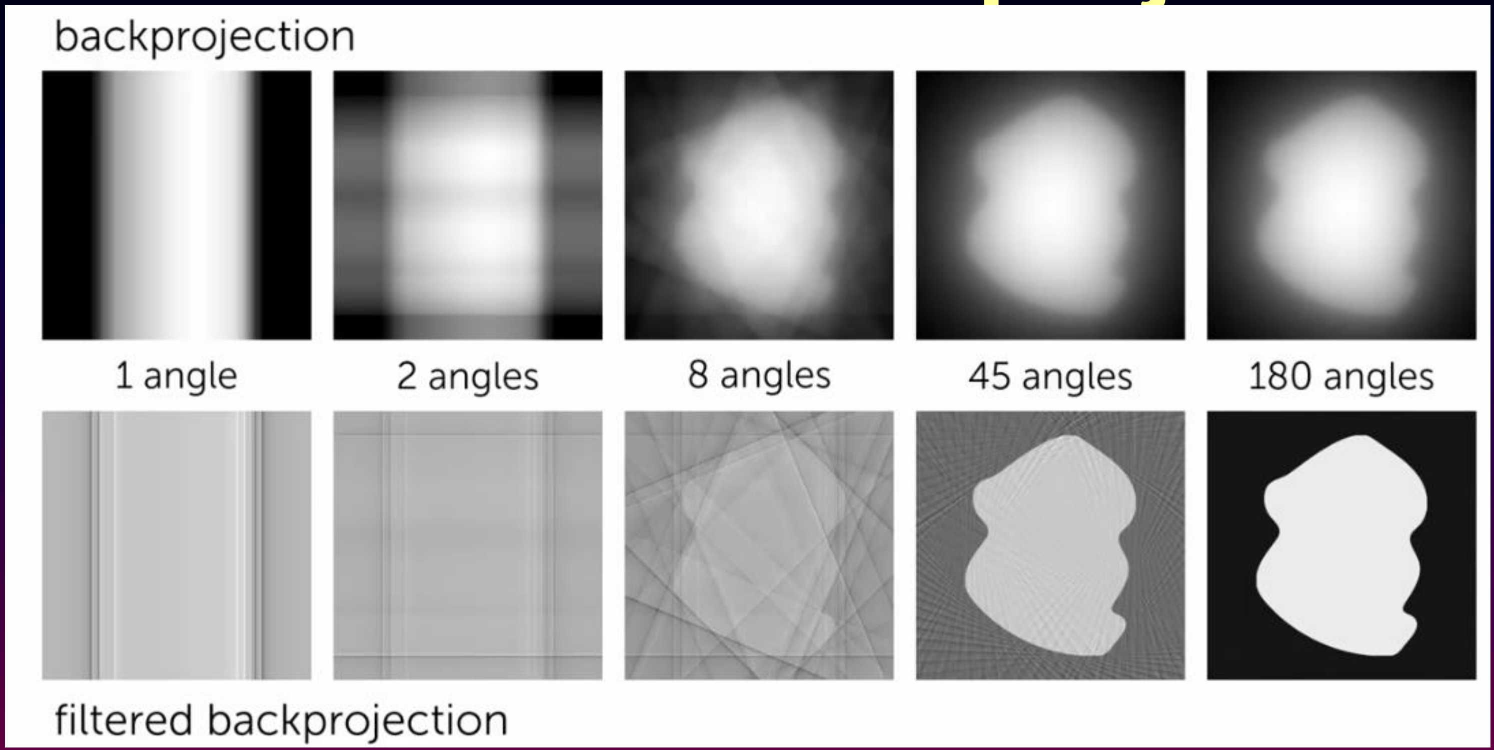


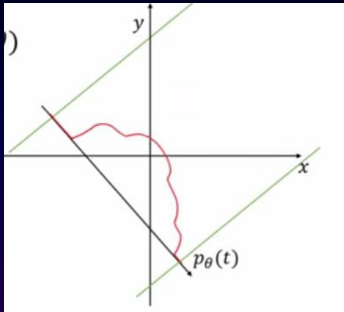
Another approach: Filtered backprojection
→…

https://en.wikipedia.org/wiki/Projection-slice_theorem

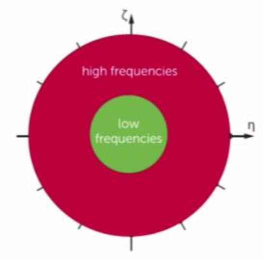
"Computed Tomography and the ASTRA Toolbox" course. University of Antwerp. September 10, 2015. http://visielab.uantwerpen.be/astra-training

Filtered Backprojection





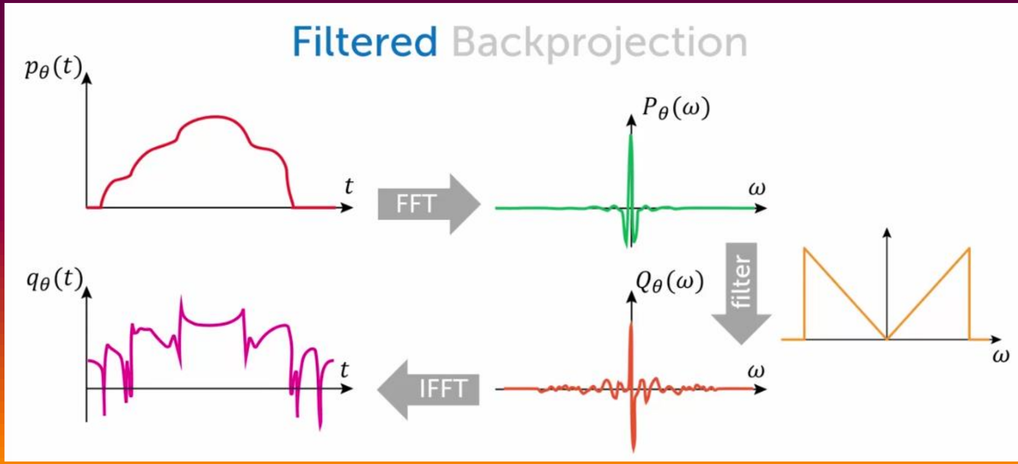
High frequencies under-represented…



$$f_{bp}(x,y) = \int p_\theta(x\cos\theta + y\sin\theta)\,d\theta$$

$$f_{fbp}(x,y) = \int q_\theta(x\cos\theta + y\sin\theta)\,d\theta$$

$$\text{with} \quad q_\theta(t) = \int P_\theta(\omega)|\omega|e^{i2\pi\omega t}d\omega$$



"Computed Tomography and the ASTRA Toolbox" course. University of Antwerp. September 10, 2015.
http://visielab.uantwerpen.be/astra-training

Gabor transform

The **Gabor transform**, named after Dennis Gabor, is a special case of the short-time Fourier transform. It is used to determine the sinusoidal frequency and phase content of local sections of a signal as it changes over time. The function to be transformed is first multiplied by a Gaussian function, which can be regarded as a window function, and the resulting function is then transformed with a Fourier transform to derive the time-frequency analysis.[1] The window function means that the signal near the time being analyzed will have higher weight. The Gabor transform of a signal x(t) is defined by this formula:

$$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

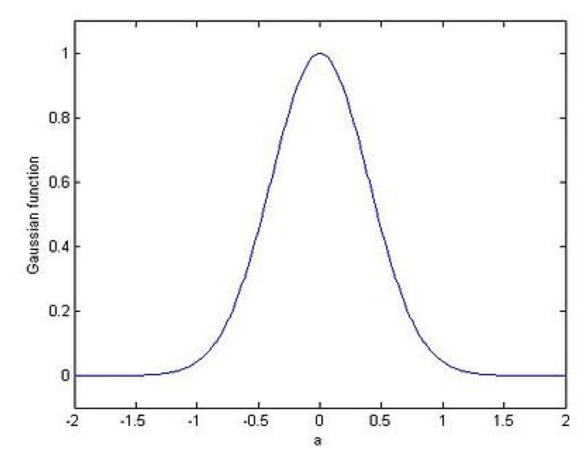
The Gaussian function has infinite range and it is impractical for implementation. However, a level of significance can be chosen (for instance 0.00001) for the distribution of the Gaussian function.

$$\begin{cases} e^{-\pi a^2} \geq 0.00001; & |a| \leq 1.9143 \\ e^{-\pi a^2} < 0.00001; & |a| > 1.9143 \end{cases}$$

Outside these limits of integration ($|a| > 1.9143$) the Gaussian function is small enough to be ignored. Thus the Gabor transform can be satisfactorily approximated as

$$G_x(t,f) = \int_{-1.9143+t}^{1.9143+t} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$$

This simplification makes the Gabor transform practical and realizable.



Magnitude of Gaussian function.

https://en.wikipedia.org/wiki/Gabor_transform
http://www.its.caltech.edu/~matilde/GaborLocalization.pdf
http://bmia.bmt.tue.nl/Education/Courses/FEV/course/pdf/Petkov_Gabor_functions2011.pdf

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Gabor transform

	Signal	Gabor transform	Remarks
	$x(t)$	$G_x(t,f) = \int_{-\infty}^{\infty} e^{-\pi(\tau-t)^2} e^{-j2\pi f\tau} x(\tau)\, d\tau$	
1	$a \cdot x(t) + b \cdot y(t)$	$a \cdot G_x(t,f) + b \cdot G_y(t,f)$	Linearity property
2	$x(t-t_0)$	$G_x(t-t_0,f)e^{-j2\pi f t_0}$	Shifting property
3	$x(t)e^{j2\pi f_0 t}$	$G_x(t,f-f_0)$	Modulation property

		Remarks
1	$\int_{-\infty}^{\infty} \lvert G_x(t,f)\rvert^2\, df = \int_{-\infty}^{\infty} e^{-2\pi(\tau-t)^2} \lvert x(\tau)\rvert^2 d\tau \approx \int_{u-1.9143}^{u+1.9143} e^{-2\pi(\tau-u)^2} \lvert x(\tau)\rvert^2 d\tau$	Power integration property
2	$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_x(t,f)G_y^*(t,f)\, df\, dt = \int_{-\infty}^{\infty} x(\tau)y^*(\tau)\, d\tau$	Energy sum property
3	$\begin{cases} \int_{-\infty}^{\infty} \lvert G_x(t,f)\rvert^2 df < e^{-2\pi(t-t_0)^2} \int_{-\infty}^{\infty} \lvert G_x(t_0,f)\rvert^2\, df; & \text{if } x(t)=0 \text{ for } t>t_0 \\ \int_{-\infty}^{\infty} \lvert G_x(t,f)\rvert^2\, dt < e^{-2\pi(f-f_0)^2} \int_{-\infty}^{\infty} \lvert G_x(t,f_0)\rvert^2\, dt; & \text{if } X(f)=FT[x(t)]=0 \text{ for } f>f_0 \end{cases}$	Power decay property
4	$\int_{-\infty}^{\infty} G_x(t,f)e^{j2\pi ktf}\, df = e^{-\pi(k-1)^2t^2} x(kt)$	Integration property
5	$\int_{-\infty}^{\infty} G_x(t,f)e^{j2\pi tf}\, df = x(t)$	Recovery property

https://en.wikipedia.org/wiki/Gabor_transform
http://www.its.caltech.edu/~matilde/GaborLocalization.pdf
http://bmia.bmt.tue.nl/Education/Courses/FEV/course/pdf/Petkov_Gabor_functions2011.pdf

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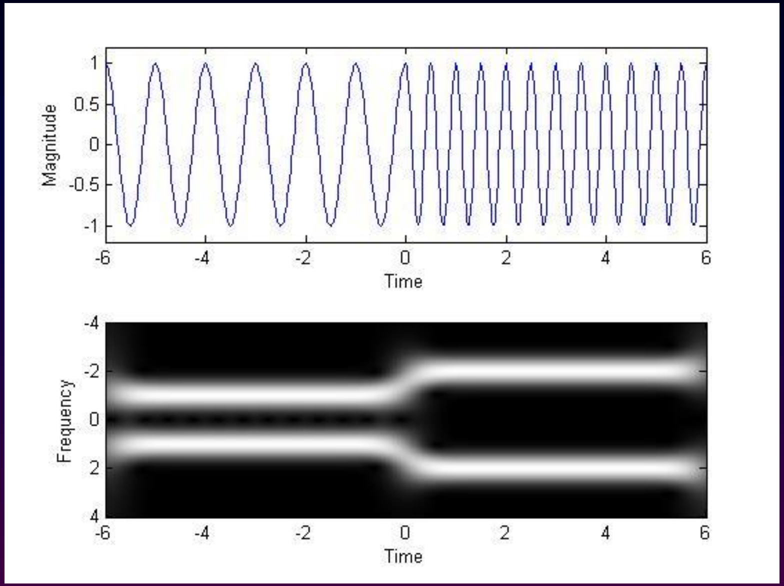
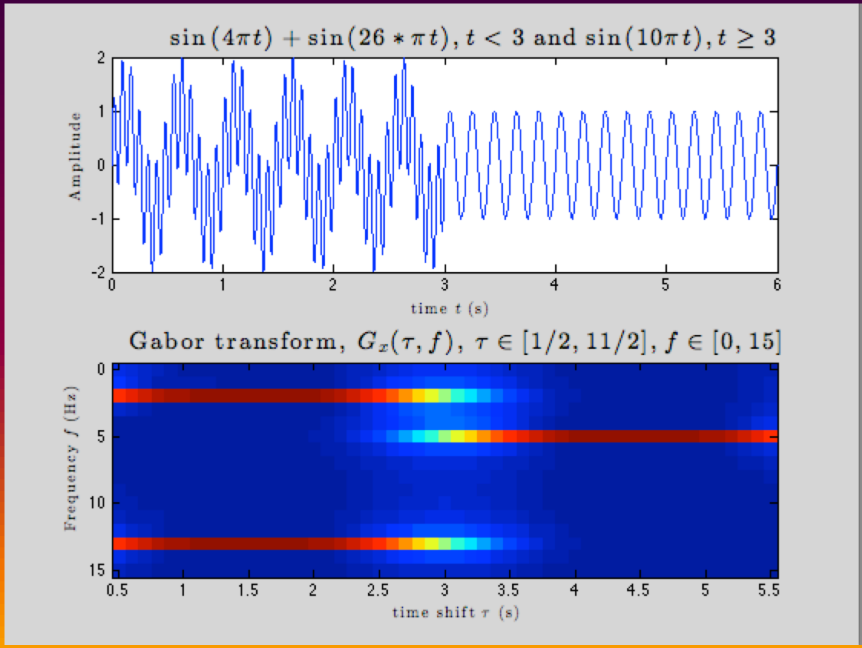
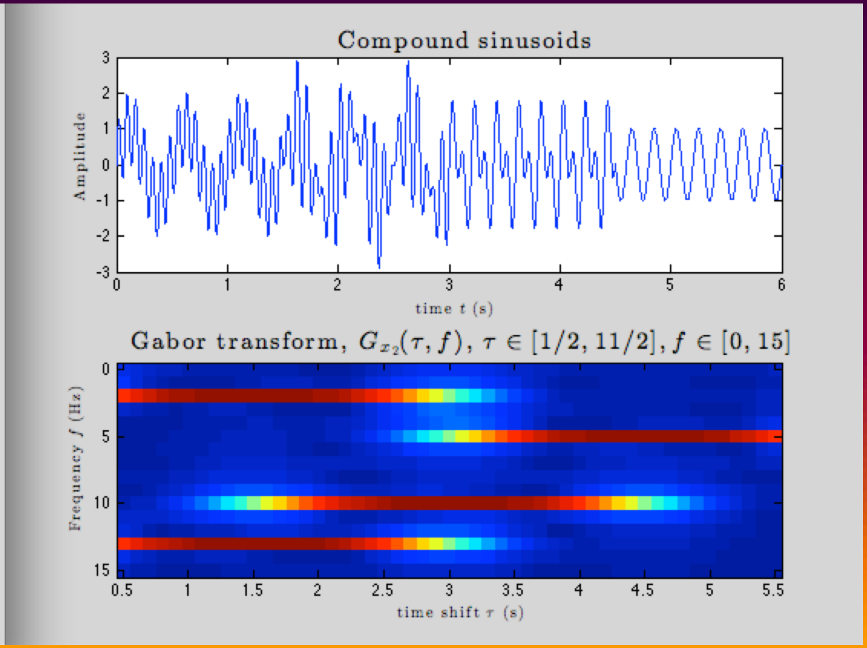
Gabor transform – 1D

$$x(t) = \begin{cases} \cos(2\pi t) & \text{for } t \leq 0, \\ \cos(4\pi t) & \text{for } t > 0. \end{cases}$$







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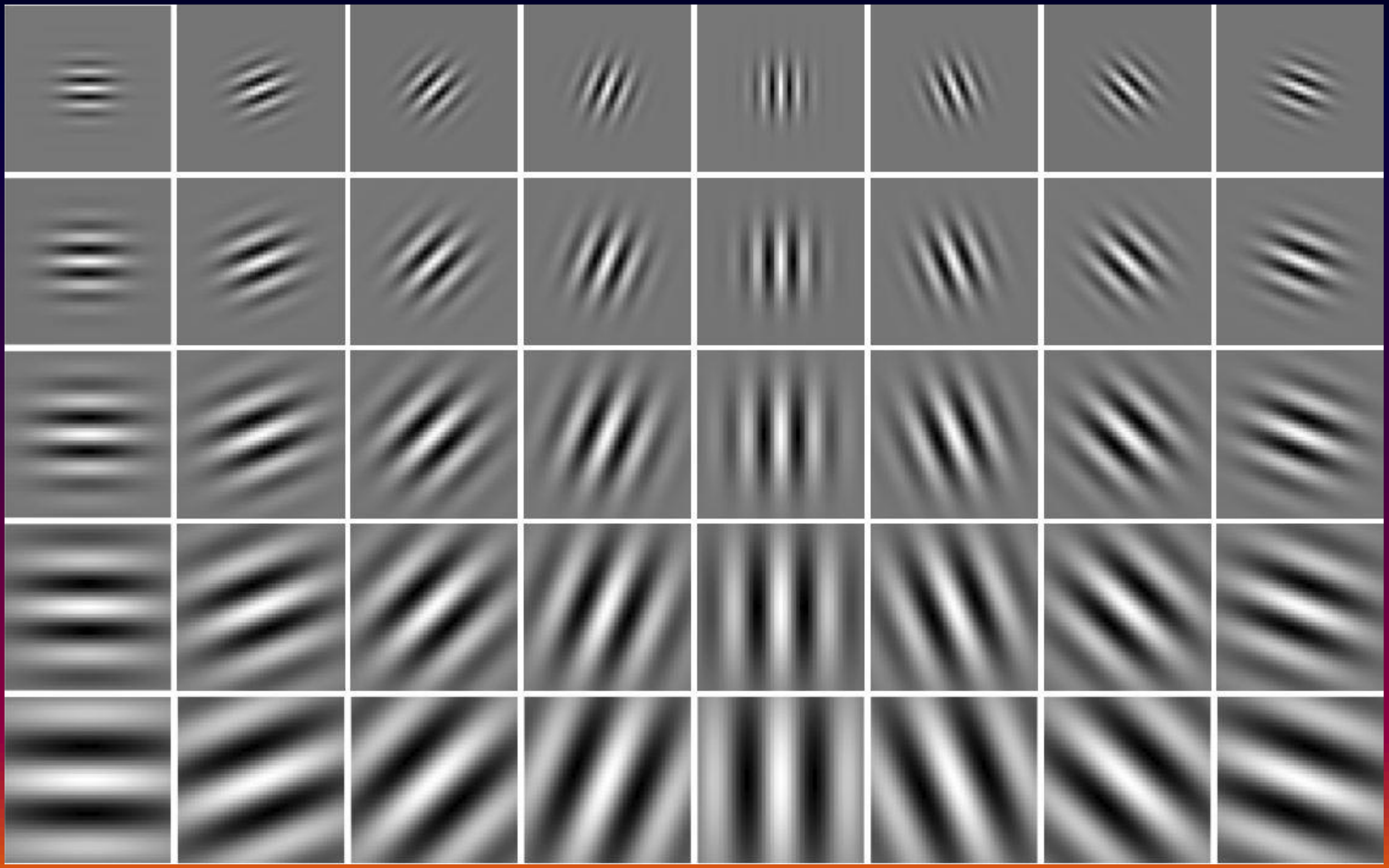
2D Gabor transform filters



By convolving an image with a set of Gabor filters, a set of image descriptors can be built, used in machine learning and classification

Mellin transform

https://en.wikipedia.org/wiki/Mellin_transform

$$M(z) \equiv \int_0^\infty dx\, f(x) x^{z-1}$$

$$M(-ju) \equiv \int_0^\infty dx\, f(x) x^{-ju-1}$$

$$M(-ju,-jv) \equiv \int_0^\infty dx \int_0^\infty dy\, f(x,y) x^{-ju-1} y^{-jv-1}$$

$$M(-ju) \equiv \int_0^\infty dx\, f(x) x^{-ju-1}$$

$$f_1(x,y) \longrightarrow M_1(-ju,-jv)$$

$$f_2(x,y) = f_1(xa, ya) \longrightarrow M_2(-ju,-jv)$$

$$\Downarrow$$

$$M_2(-ju,-jv) = a^{-ju-jv} M_1(-ju,-jv)$$

SCALE INVARIANCE

$$m(u) \equiv M(-ju) = \int_0^\infty dx\, f(x) x^{-ju-1}$$

$$x = e^{\xi} \quad \Leftrightarrow \quad \xi = \ln(x)$$

$$g(\xi) = f(e^{\xi})$$

$$\Downarrow$$

$$m(u) = \int_{-\infty}^{\infty} g(\xi) e^{-ju\xi} d\xi$$

Mellin Transform of the function $f(x)$ is just the Fourier Transform of the function $g(\xi) = f(e^{\xi})$

Using Mellin-Transform for Shift, Rotation and Scale Invariant Image Recognition

https://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=2ahUKEwicsZfWveLdAhVUfMAKHbbLCOsQFjABegQICBAC&url=http%3A%2F%2Flapt.ece.northwestern.edu%2Ffiles%2Fmellin_transform_correlator.pdf&usg=AOvVaw1KquMNCEOGAXQIpPAXsHU1